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Complex conformal field theory in non-Hermitian systems

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Phys. Rev. B 112, 085112 (2025)
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Complex conformal field theory (CFT)

new type of CFT with complex conformal data

(central charges, scaling dimensions, …)

– Unique critical phenomena in non-Hermitian systems

(open quantum many-body systems)

– Universal descriptions of discontinuous phase transitions hidden in Hermitian systems

Phys. Rev. B 112, 085112 (2025)
Phys. Rev. B 114, 115112 (2026)
arXiv:2606.16785

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Outline

1. Introduction

2. Complex conformal field theory (review)

3. Complex entanglement entropy for complex conformal field theory

4. Complex nonlinear sigma models

Non-Hermitian physics

Despite the enormous success, much of conventional many-body theory is formulated for **Hermitian systems at equilibrium**.

→ **Richer properties appear in non-Hermitian systems!**

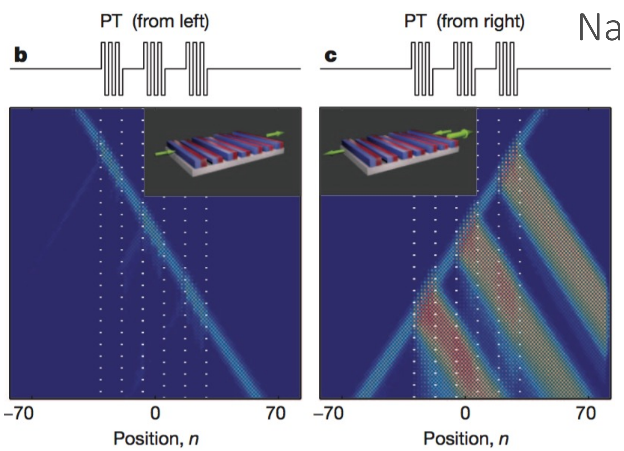
☆ Non-Hermiticity arises from **dissipation**, i.e., exchanges of energy or particles with an environment.

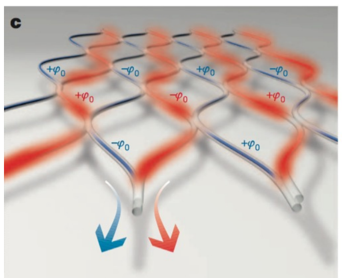
El-Ganainy *et al.*, Nat. Phys. **14**, 11 (2018)

- **Photonic lattices with gain/loss**

Unidirectional light transport

Regensburger *et al.*, Nature **488**, 167 (2012)

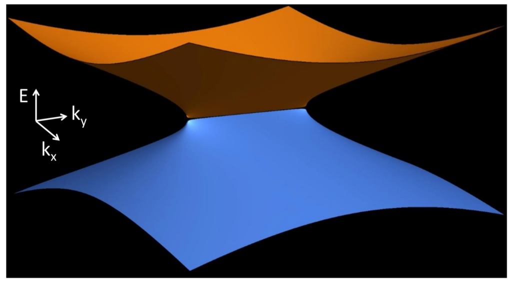


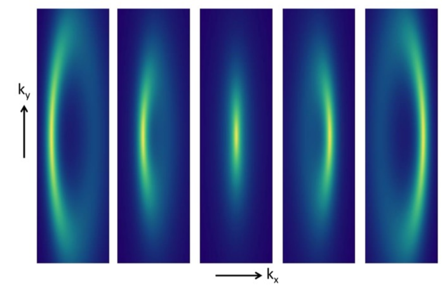


- **Finite-lifetime quasiparticles**

Bulk Fermi arc due to non-Hermitian self-energy

Kozii & Fu, arXiv: 1708.05841

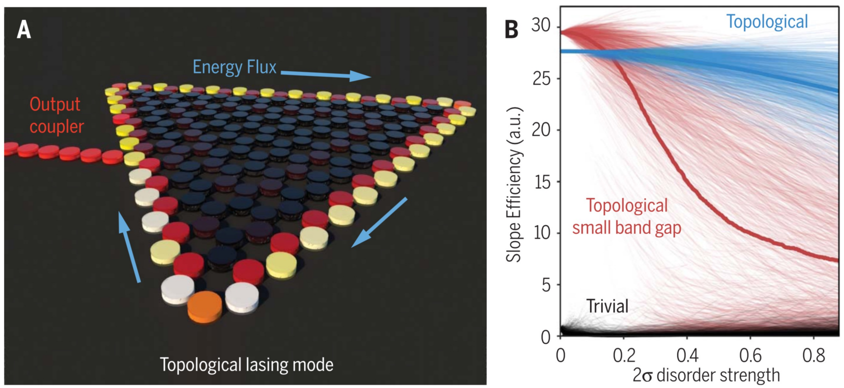




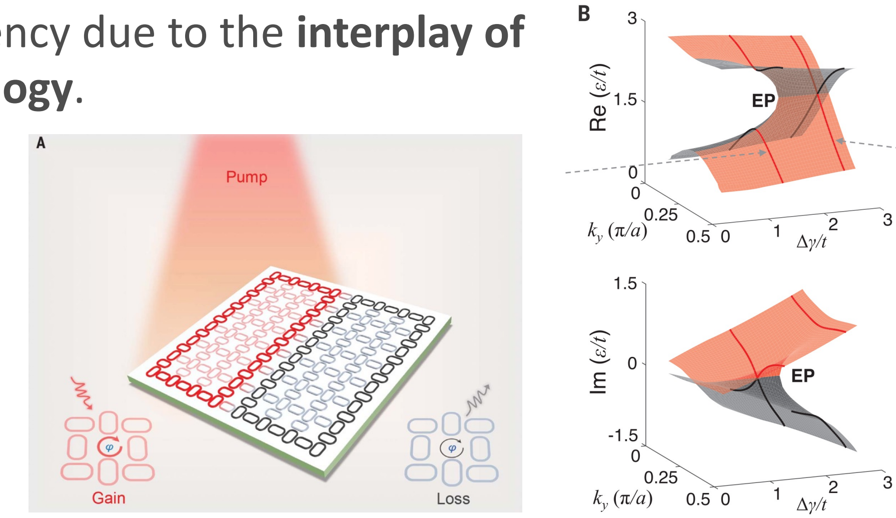
Non-Hermitian topological systems

- **Topological laser**

New laser with high efficiency due to the **interplay of non-Hermiticity and topology**.



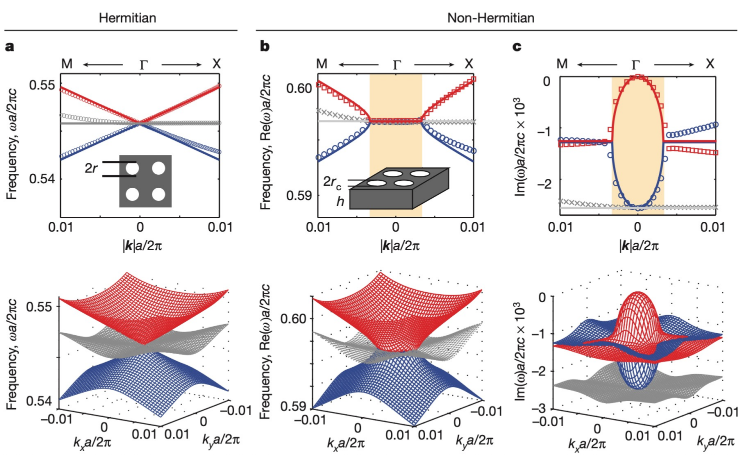
Bandres *et al.*, Science **359**, eaar4005 (2018)



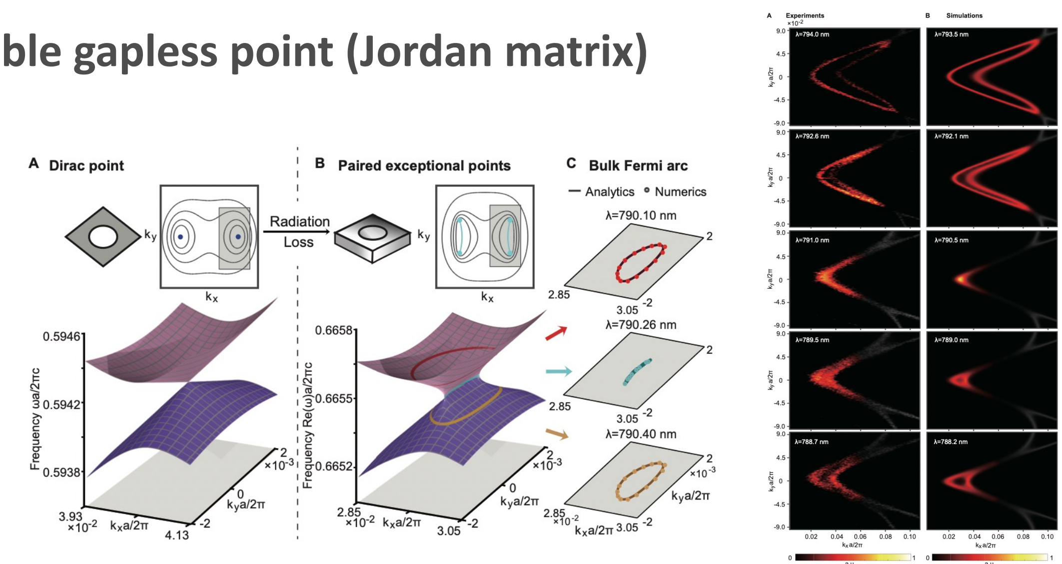
Zhao *et al.*, Science **365**, 1163 (2019)

- **Exceptional point**

Nondiagonalizable gapless point (Jordan matrix)



Zhen *et al.*, Nature **525**, 354 (2015)



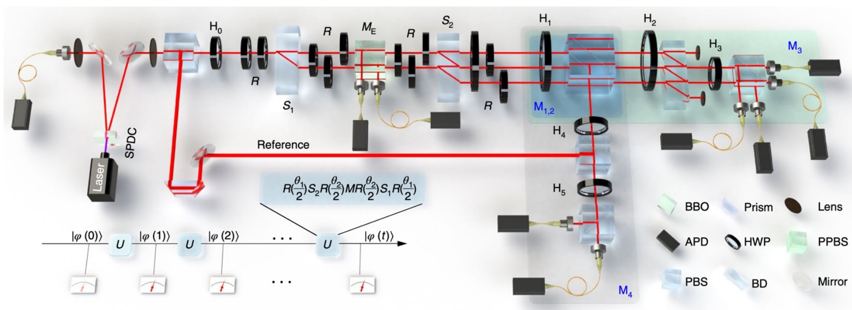
Zhou *et al.*, Science **359**, 1009 (2018)

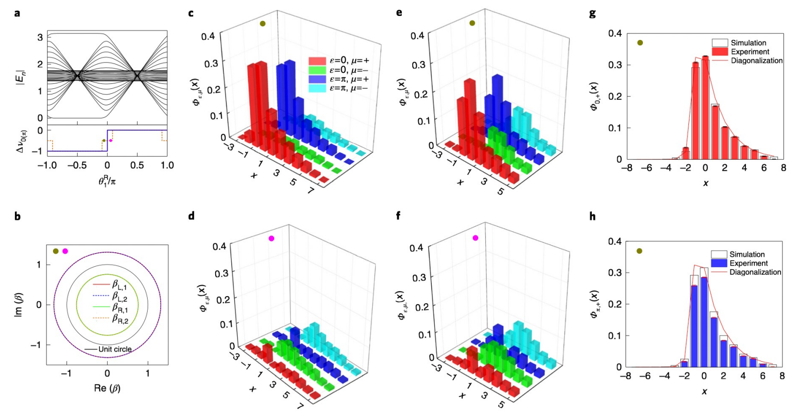
Non-Hermitian quantum physics (1)

Unique non-Hermitian phenomena have been observed in recent quantum experiments.

- **Quantum walk (single photons)**

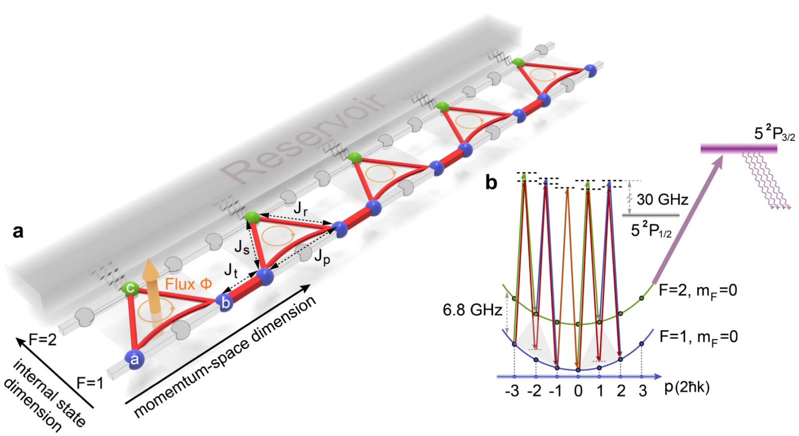
Xiao *et al.*, Nat. Phys. **16**, 761 (2020)

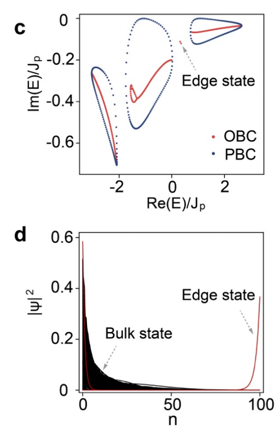


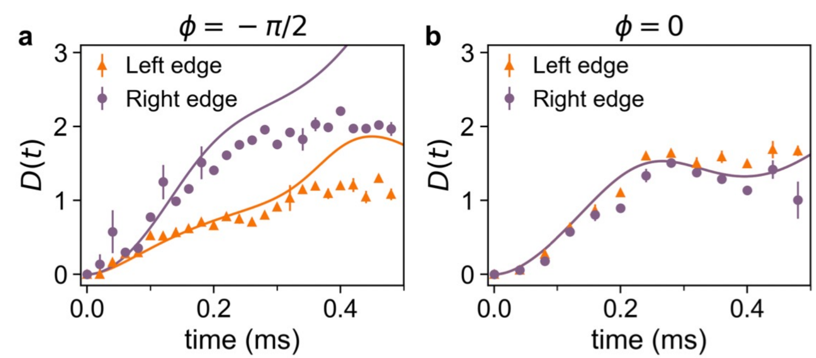


- **Ultracold atoms**

Liang *et al.*, PRL **129**, 070401 (2022)







Non-Hermitian quantum physics (2)

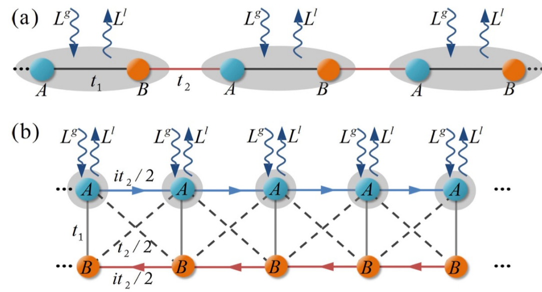
Non-Hermitian operators are relevant even to more generic open quantum systems that are not characterized by Hamiltonians.

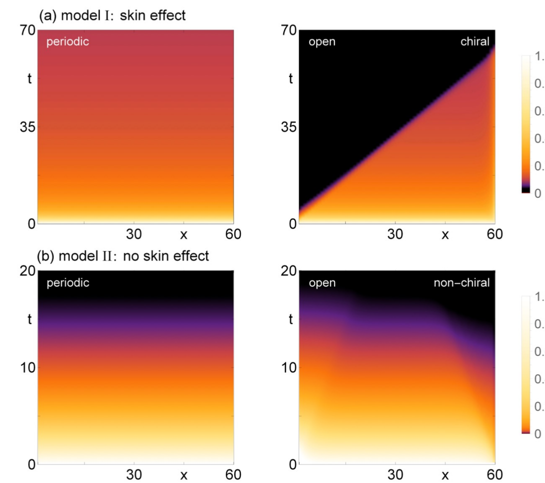
Song, Yao & Wang, PRL **123**, 170401 (2019)

- **Master equation**

$$d\hat{\rho}/dt = \underline{\mathcal{L}}\hat{\rho}$$

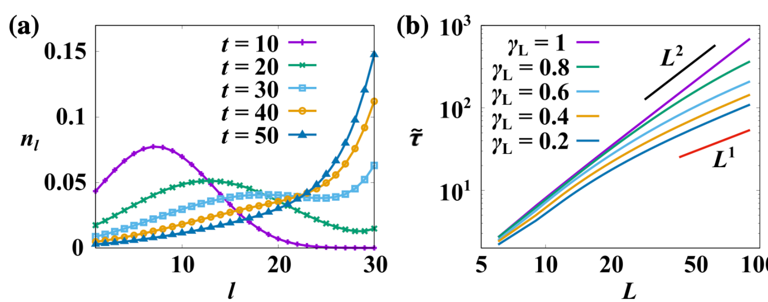
Non-Hermitian superoperator (Liouvillian)

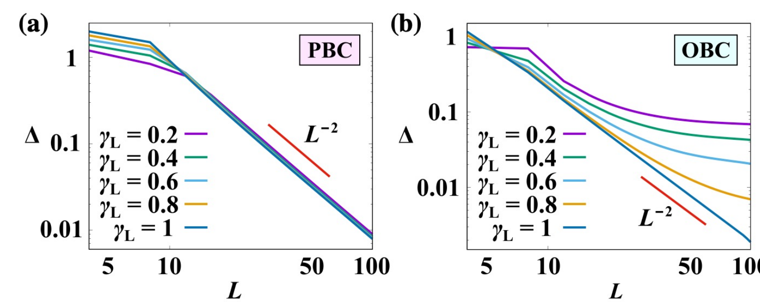




Chiral damping due to the skin effect

Slowdown of relaxation due to the skin effect





Haga *et al.*, PRL **127**, 070402 (2021); Mori *et al.*, PRL **125**, 230604 (2020)

Non-Hermitian quantum criticality

☆ Non-Hermiticity leads to unique critical phenomena that have no counterparts in Hermitian systems.

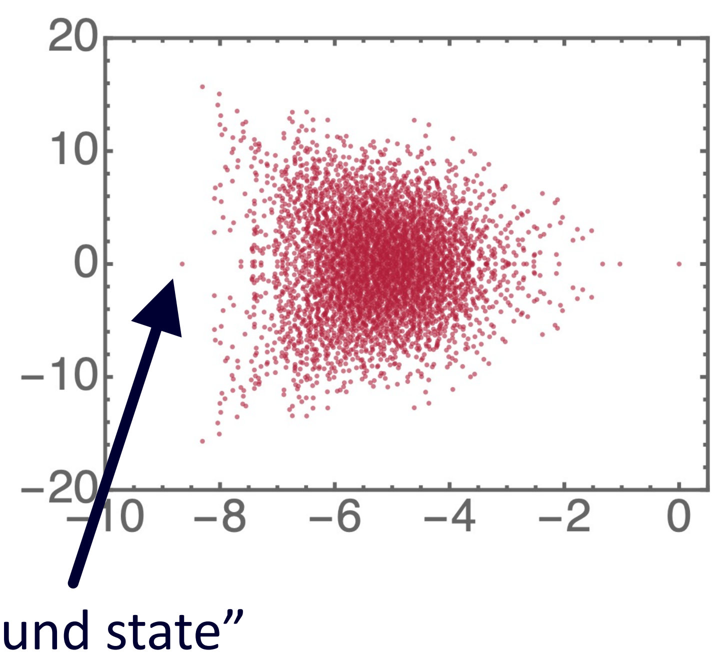
$$\hat{H}\,\underbrace{|\psi_{\mathrm{GS}}\rangle}_{\text{right eigenstate}} = E_{\mathrm{GS}}\,|\psi_{\mathrm{GS}}\rangle\,, \qquad \hat{H}^{\dagger}\underbrace{|\psi_{\mathrm{GS}}\rangle\!\rangle}_{\text{left eigenstate}} = E_{\mathrm{GS}}^{*}|\psi_{\mathrm{GS}}\rangle\!\rangle$$

– Complex eigenenergy

$$E_{\mathrm{GS}} \in \mathbb{C}$$

– Different right and left eigenstates

$$|\psi_{\mathrm{GS}}\rangle \neq |\psi_{\mathrm{GS}}\rangle\!\rangle$$

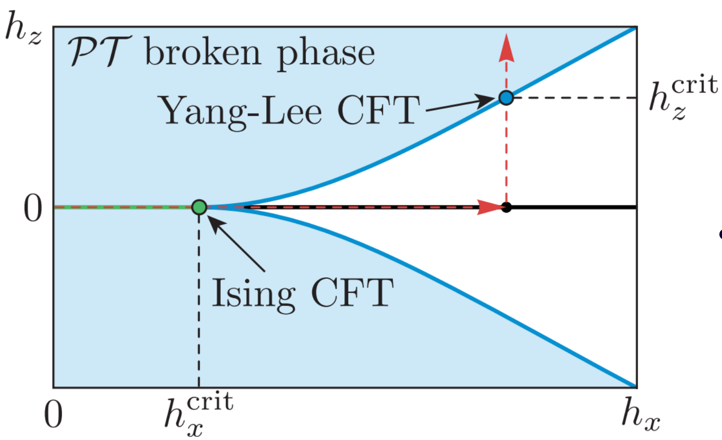


Yang-Lee edge singularity (1)

☆ Yang-Lee edge singularity

$$\hat{H}_{\mathrm{YL}} = -\sum_{n=1}^{N}\left(J\hat{Z}_n\hat{Z}_{n+1} + h_x\hat{X}_n + \mathrm{i}h_z\hat{Z}_n\right)$$

Ising model with an imaginary longitudinal magnetic field



Arguello Cruz *et al.*,
PRX **16**, 011022 (2026)

$|h_z| < h_z^{\mathrm{crit}}$: real spectrum

$|h_z| > h_z^{\mathrm{crit}}$: complex spectrum

→ $|h_z| = h_z^{\mathrm{crit}}$ gives a critical point unique to non-Hermitian systems

– spontaneous breaking of PT symmetry

– exceptional point

Bender & Boettcher,
PRL **80**, 5243 (1998)

Yang-Lee edge singularity (2)

– Originally introduced to rigorously show the presence of phase transitions in Hermitian systems (“Yang-Lee zeros”)

Yang & Lee, PR **87**, 404 & 410 (1952)

– Landau-Ginzburg theory

$$S = \int d^d\boldsymbol{r} \left[\frac{1}{2} (\nabla\phi)^2 + \mathrm{i}\gamma\phi^3 \right]$$

Fisher, PRL **40**, 1610 (1978)
Katsevich *et al.*, PRL **136**, 111602 (2026)

– 2D conformal field theory

Cardy, PRL **54**, 1354 (1985)

$$c = -\frac{22}{5} < 0$$

negative central charge!

One of the simplest minimal models M (2, 5)

– Higher-dimensional generalizations

Fan *et al.*, arXiv:2505.06342
Arguello Cruz *et al.*, PRX **16**, 011022 (2026)
Miro & Delouche, JHEP **2025**, 37

☆ **A classic example of nonunitary critical phenomena**

Complex conformal field theory (1)

Ordinary unitary conformal field theory

Central charge is positive $c \in \mathbb{R}, \quad c > 0$

Yang-Lee conformal field theory

Central charge is real but negative $c \in \mathbb{R}, \quad c < 0$

Nonunitary, but real conformal field theory

Central charge can even be complex (nonreal)!

$$c \in \mathbb{C} \quad (c \notin \mathbb{R})$$

“Complex conformal field theory”

Complex conformal field theory (2)

Complex conformal field theory (CFT)

Gorbenko *et al.*, JHEP **2018**, 108

new type of CFT with complex conformal data (central charges, scaling dimensions, …)

– Dates back to zeros of partition functions in the complex temperature plane

Fisher (1965)

– Unique critical phenomena in non-Hermitian systems (open quantum many-body systems)

– Universal descriptions of **discontinuous phase transitions** hidden in Hermitian systems

(e.g., Potts model, 4D gauge theory, deconfined criticality)

– Relevance to dS/CFT correspondence?

Strominger, JHEP **2001**, 29

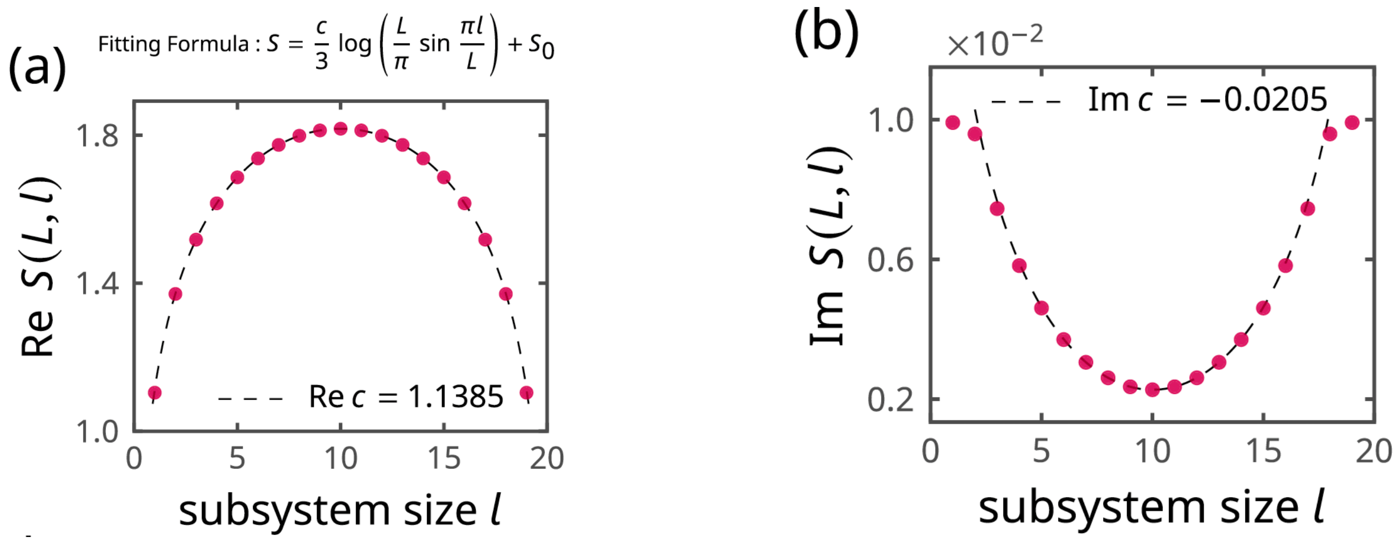
Results (1)

Shimizu & **Kawabata**,
PRB **112**, 085112 (2025)

We show that the complex-valued entanglement entropy correctly captures the complex-valued central charge!

We employ non-Hermitian reduced density matrices constructed from the combination of right and left ground states.

Applying DMRG, we calculate the complex entanglement entropy of the non-Hermitian Potts model and confirm the complex CFT.

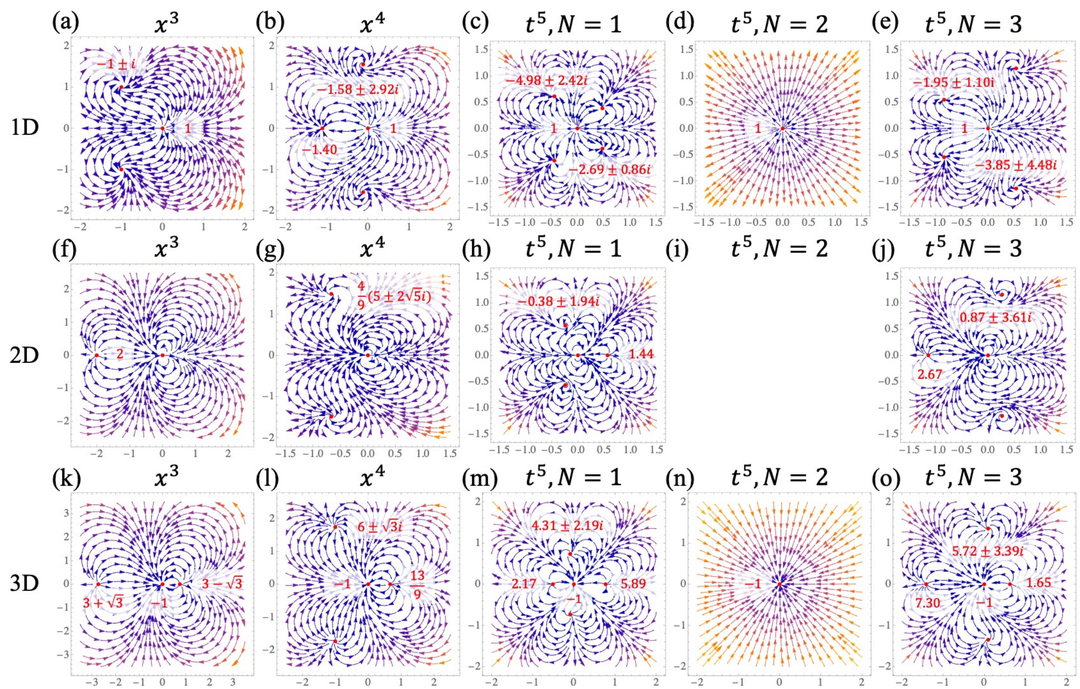


Results (2)

Yamamoto & **Kawabata**,
PRB **114**, 115112 (2026)

We study nonlinear sigma models with complexified couplings as a prototypical framework of complex CFT.

Using the perturbative renormalization-group analysis, we show that fixed points with complex scaling dimensions arise generically.



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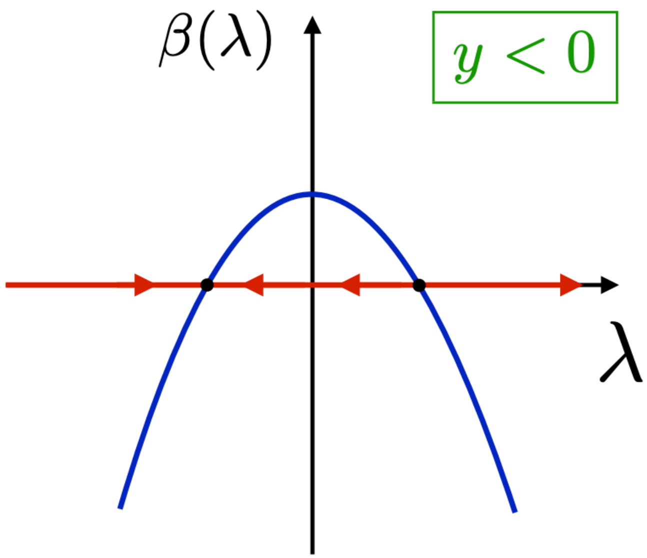
Walking (1)

Gorbenko *et al.*, JHEP **2018**, 108

Let us consider a simple renormalization group equation:

$$\frac{d\lambda}{dl} = \beta(\lambda) = -y - \lambda^2 \quad (y \in \mathbb{R}, \lambda \in \mathbb{R})$$



$\underline{y < 0}$

fixed points $\beta(\lambda_c) = 0$

$$\longrightarrow \quad \lambda_c = \pm\sqrt{-y}$$

scaling dimensions

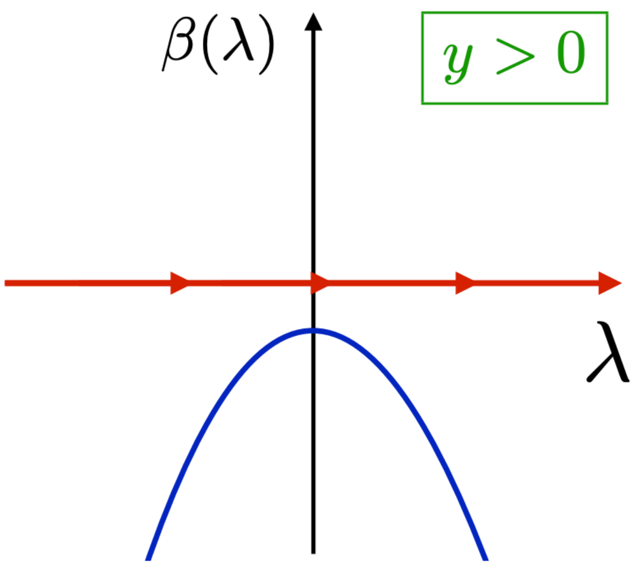
$$\begin{aligned} \Delta_\pm &= d + \beta'(\lambda_c) \\ &= d \mp 2\sqrt{-y} \end{aligned}$$

Walking (2)

Gorbenko *et al.*, JHEP **2018**, 108

Let us consider a simple renormalization group equation:

$$\frac{d\lambda}{dl} = \beta(\lambda) = -y - \lambda^2 \quad (y \in \mathbb{R}, \lambda \in \mathbb{R})$$



$\underline{y > 0}$ $\quad \beta < 0$ for all $\lambda \in \mathbb{R}$

→ no fixed points

However, the RG flow slows down for $0 < y \ll 1$

$$\Delta l \sim -\int_{-1}^{1} \frac{d\lambda}{\beta(\lambda)} \sim \frac{\pi}{\sqrt{y}}$$

$$\xi/a \sim e^{\Delta l} \sim e^{\pi/\sqrt{y}}$$

☆ **Despite the absence of fixed points, it behaves as if it were critical.**

"walking RG flow"

Walking and complex criticality

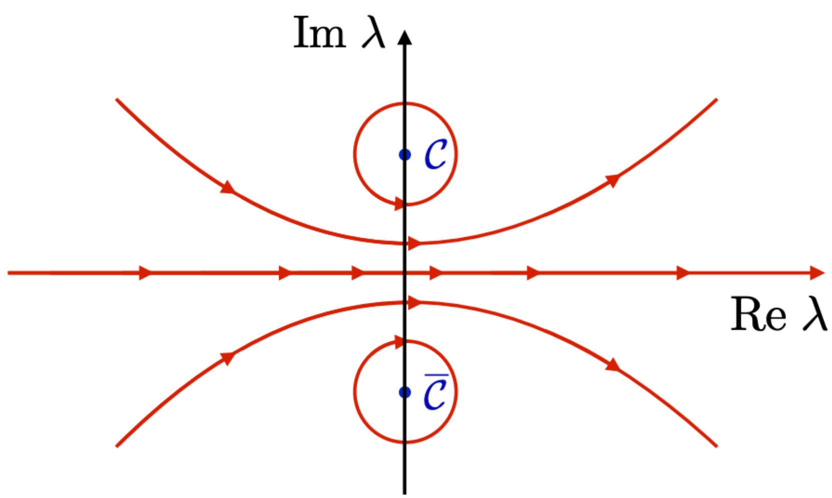
Gorbenko *et al.*, JHEP **2018**, 108

☆ **Walking RG flow can be captured by complex fixed points.**

$$\frac{d\lambda}{dl} = \beta(\lambda) = -y - \lambda^2 \quad (y \in \mathbb{R}, \underline{\lambda \in \mathbb{C}})$$

$$\lambda_c = \begin{cases} \pm\sqrt{-y} \in \mathbb{R} & (y < 0) \\ \pm i\sqrt{y} \in i\mathbb{R} & (y > 0) \end{cases}$$

Fixed points appear in the complex-λ plane.



$$\Delta_\pm = d + \beta'(\lambda_c) = d \mp 2i\sqrt{y} \in \mathbb{C}$$

complex scaling dimensions

Small imaginary parts lead to walking and weakly discontinuous behavior.
(gapped, but nearly critical)

Potts model

Simple example of weakly discontinuous phase transitions:
(walking RG flow)

Potts model (*Q*-state generalization of the Ising model)

[classical]

$$H[\{s_i\}] = -\beta \sum_{\langle i,j \rangle} \underline{\delta_{s_i,s_j}}$$

Q-state classical spins

Potts, Math. Proc. Camb. Philos. Soc. **48**, 106 (1952)

[quantum (1+1 D)]

$$\hat{H}_0 = -\sum_{i=1}^{L} \sum_{k=1}^{Q-1} \left[J \left(\hat{\sigma}_i^\dagger \hat{\sigma}_{i+1} \right)^k + h \hat{\tau}_i^k \right]$$

$J > h$ ordered

$J < h$ disordered

$J = h$ phase transition (Kramers-Wannier duality point)

Continuous phase transitions

☆**Types of phase transitions depend on *Q***

$\underline{Q \leq 4}$

continuous phase transitions

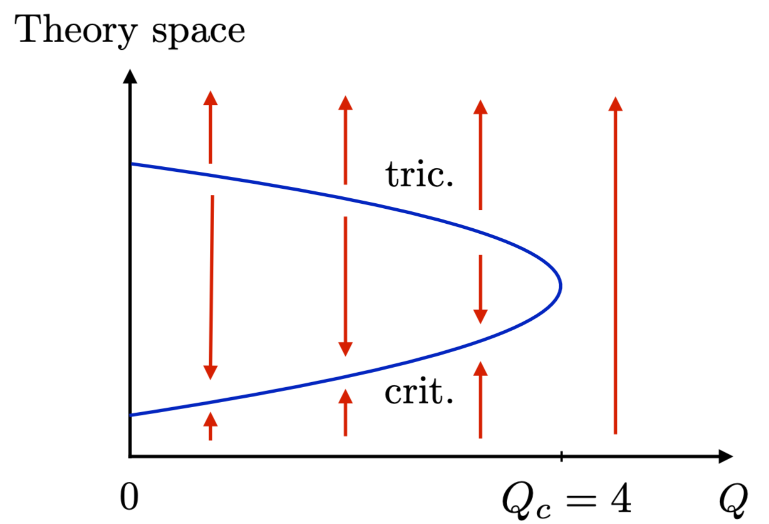
described by (unitary) CFT

$$c = 1 - \frac{3}{\pi}\frac{\left[\cos^{-1}(Q/2-1)\right]^2}{2\pi - \cos^{-1}(Q/2-1)}$$

$$Q = 2: \quad c = \frac{1}{2} \quad \text{(Ising)}$$

$$Q = 3: \quad c = \frac{4}{5}$$

$$Q = 4: \quad c = 1$$



Gorbenko *et al.*, SciPost Phys. **5**, 050 (2018)

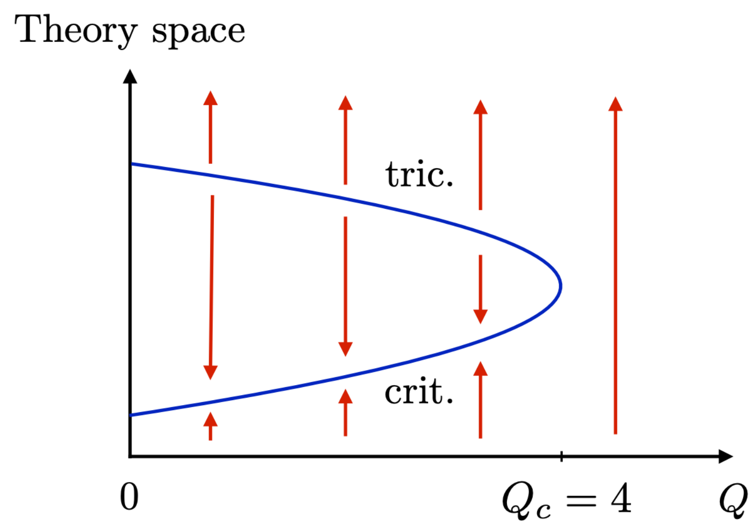
Weakly discontinuous phase transitions

$\underline{Q > 4}$

<u>discontinuous</u> phase transitions

finite correlation length $\xi < \infty$

(continuous ones: $\xi \to \infty$)



Gorbenko *et al.*, SciPost Phys. **5**, 050 (2018)

☆**For *Q* = 5, the discontinuous phase transition almost behaves as a continuous phase transition.**

finite but large correlation length $\xi \propto \exp\left(\frac{\pi^2}{\sqrt{Q-4}}\right)$

Q	5	6	7	8	9	10
ξ	2512.2	158.9	48.1	23.9	14.9	10.6

Buffenoir & Wallon, J. Phys. A **26**, 3045 (1993)

We may not judge continuous or discontinuous phase transitions even for very large systems!

Complex conformal field theory

☆ **Discontinuous phase transitions of the Potts model can be connected to nearby complex fixed points!**

Gorbenko *et al.*, SciPost Phys. **5**, 050 (2018)

Analytically continue conformal data to $Q > 4$

$$c = 1 + \frac{3}{\pi} \frac{\left[\cosh^{-1}(Q/2-1)\right]^2}{2\pi \pm \mathrm{i}\cosh^{-1}(Q/2-1)} \in \begin{cases} \mathbb{R} & (Q \le 4) \\ \mathbb{C} & (Q > 4) \end{cases}$$

$$c \approx 1.138 \pm \underbrace{0.021\mathrm{i}}_{\text{tiny imaginary part}} \quad (Q = 5)$$

Because of tiny imaginary parts, the complex fixed points influence the RG flow even on the real-coupling plane.

→ **weakly discontinuous phase transition (almost continuous)**

Universal discontinuous phase transitions

Traditionally, continuous phase transitions have attracted special attention because they exhibit universality associated with scale/conformal invariance.

Discontinuous phase transitions are considered to be less universal.

→ **Discontinuous phase transitions can be governed by nearby complex CFT.**

Complexification can reveal hidden universality of discontinuous phase transitions

To directly reach complex fixed points, we need to add non-Hermitian perturbations.

Questions

Complex CFT describes unique critical phenomena that have no counterparts in Hermitian systems.

(1) Many basic properties of complex CFT are unclear.

(because of nonunitary and irrational nature)

(2) How can we realize complex CFT in microscopic lattice models?

Can we extract conformal data numerically?

(3) More physical applications?

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Non-Hermitian Potts model

Can we explicitly realize complex fixed points in microscopic lattice models beyond the field-theory analysis?

☆ **Non-Hermitian Potts model**

Jacobsen & Wiese, PRL **133**, 077101 (2024)
Tang *et al.*, PRL **133**, 076504 (2024)

$$\hat{H} = \hat{H}_0 + \hat{H}_1$$

$$\hat{H}_0 = -\sum_{i=1}^{L}\sum_{k=1}^{Q-1}\left[J\left(\hat{\sigma}_i^{\dagger}\hat{\sigma}_{i+1}\right)^k + h\hat{\tau}_i^k\right]$$

$$\hat{H}_1 = \lambda\sum_{i=1}^{L}\sum_{k_1,k_2=1}^{Q-1}\left[\left(\hat{\tau}_i^{k_1} + \hat{\tau}_{i+1}^{k_1}\right)\left(\hat{\sigma}_i^{\dagger}\hat{\sigma}_{i+1}\right)^{k_2} + \left(\hat{\sigma}_i^{\dagger}\hat{\sigma}_{i+1}\right)^{k_1}\left(\hat{\tau}_i^{k_2} + \hat{\tau}_{i+1}^{k_2}\right)\right]$$

non-Hermitian perturbation (λ: complex)

critical point: $J_{\mathrm{c}} = h_{\mathrm{c}} = 1, \quad \lambda_{\mathrm{c}} \approx 0.079 \pm 0.060\mathrm{i} \quad (Q=5)$

(The non-Hermitian perturbation is chosen such that it satisfies the Kramers-Wannier duality)

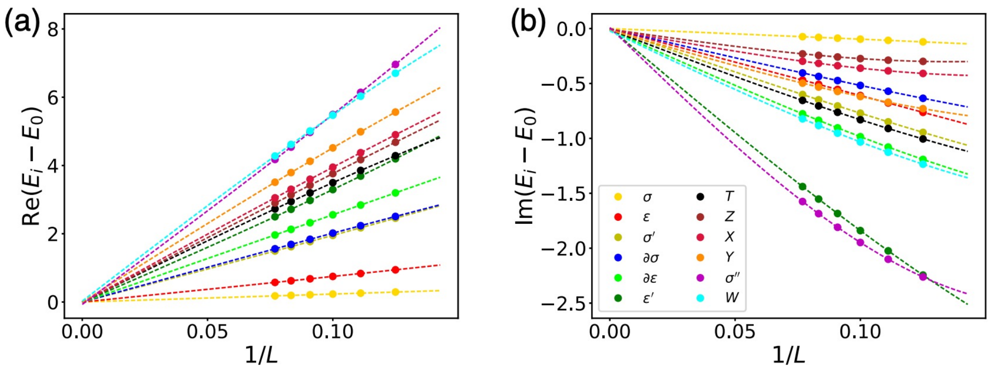
Complex scaling dimensions

CFT scaling of eigenenergy:

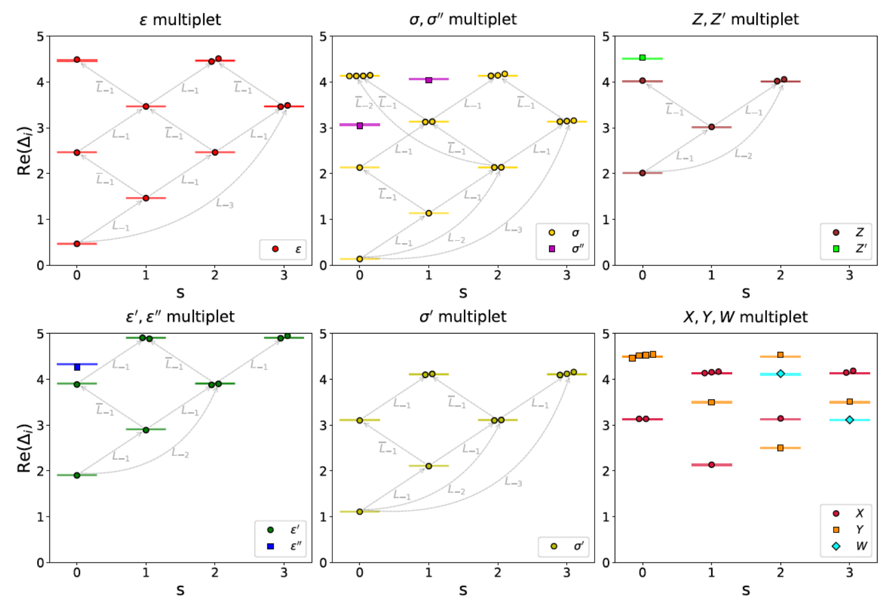
$$\underline{E_n - E_0} = \frac{2\pi v}{L}\underline{\Delta_n} + \cdots$$

complex eigenenergy **complex scaling dimensions**

Numerical demonstration of complex CFT



Tang *et al.*, PRL **133**, 076504 (2024)



Quantum entanglement

CFT leads to the universal scaling of entanglement entropy

$$S = \frac{c}{3} \log l$$

Holzhey, Larsen & Wilczek, Nucl. Phys. B **424**, 443 (1994)

Calabrese & Cardy, J. Stat. Mech. **2004**, P06002

Entanglement gives a direct numerical access to the central charge

$$S = \begin{cases} \frac{c}{3} \log \left(\frac{L}{\pi a} \sin \frac{\pi l}{L} \right) & \text{(PBC)} \\ \frac{c}{6} \log \left(\frac{2L}{\pi a} \sin \frac{\pi l}{L} \right) & \text{(OBC)} \end{cases}$$

(practically relevant, because entanglement entropy is dimensionless, and does not require fitting a nonuniversal velocity)

Physically relevant to the holographic principle

Ryu & Takayanagi, PRL **96**, 181602 (2006)

Motivation

Can entanglement still capture complex conformal field theory?

$$S = \underline{\frac{c}{3}} \log l$$

complex

Is entanglement entropy also complex?

Should we modify the formula while keeping entropy real?

Or should we generalize entropy to a complex-valued quantity?

Complex-valued entanglement entropy

☆ **Right and left eigenstates are different in non-Hermitian systems.**

$$\hat{H}\,\underbrace{|\psi_{\mathrm{GS}}\rangle}_{\text{right ground state}} = E_{\mathrm{GS}}\,|\psi_{\mathrm{GS}}\rangle\,, \qquad \hat{H}^{\dagger}\underbrace{|\psi_{\mathrm{GS}}\rangle\!\rangle}_{\text{left ground state}} = E_{\mathrm{GS}}^{*}|\psi_{\mathrm{GS}}\rangle\!\rangle$$

→ **non-Hermitian density matrix** $\dfrac{|\psi_{\mathrm{GS}}\rangle\,\langle\!\langle\psi_{\mathrm{GS}}|}{\langle\!\langle\psi_{\mathrm{GS}}|\psi_{\mathrm{GS}}\rangle}$

entanglement entropy can be complex

$$S := -\mathrm{tr}_{\mathrm{A}}\left(\hat{\rho}_{\mathrm{A}}^{\mathrm{RL}}\log\hat{\rho}_{\mathrm{A}}^{\mathrm{RL}}\right), \quad \hat{\rho}_{\mathrm{A}}^{\mathrm{RL}} := \mathrm{tr}_{\mathrm{B}}\frac{|\psi_{\mathrm{GS}}\rangle\langle\!\langle\psi_{\mathrm{GS}}|}{\langle\!\langle\psi_{\mathrm{GS}}|\psi_{\mathrm{GS}}\rangle}$$

Equivalent to pseudo-entropy in the high-energy context

Nakata *et al.*, PRD **103**, 026005 (2021)

Non-Hermitian DMRG

We apply the generalized DMRG (density matrix renormalization group) to the non-Hermitian Potts model to obtain the right and left ground states.

apply DMRG to $\hat{H}$ $\longrightarrow$ right ground state $|\psi_{\mathrm{GS}}\rangle$

apply DMRG to $\hat{H}^{\dagger}$ $\longrightarrow$ left ground state $|\psi_{\mathrm{GS}}\rangle\rangle$

We then construct the non-Hermitian density matrix $\dfrac{|\psi_{\mathrm{GS}}\rangle\langle\langle\psi_{\mathrm{GS}}|}{\langle\langle\psi_{\mathrm{GS}}|\psi_{\mathrm{GS}}\rangle}$

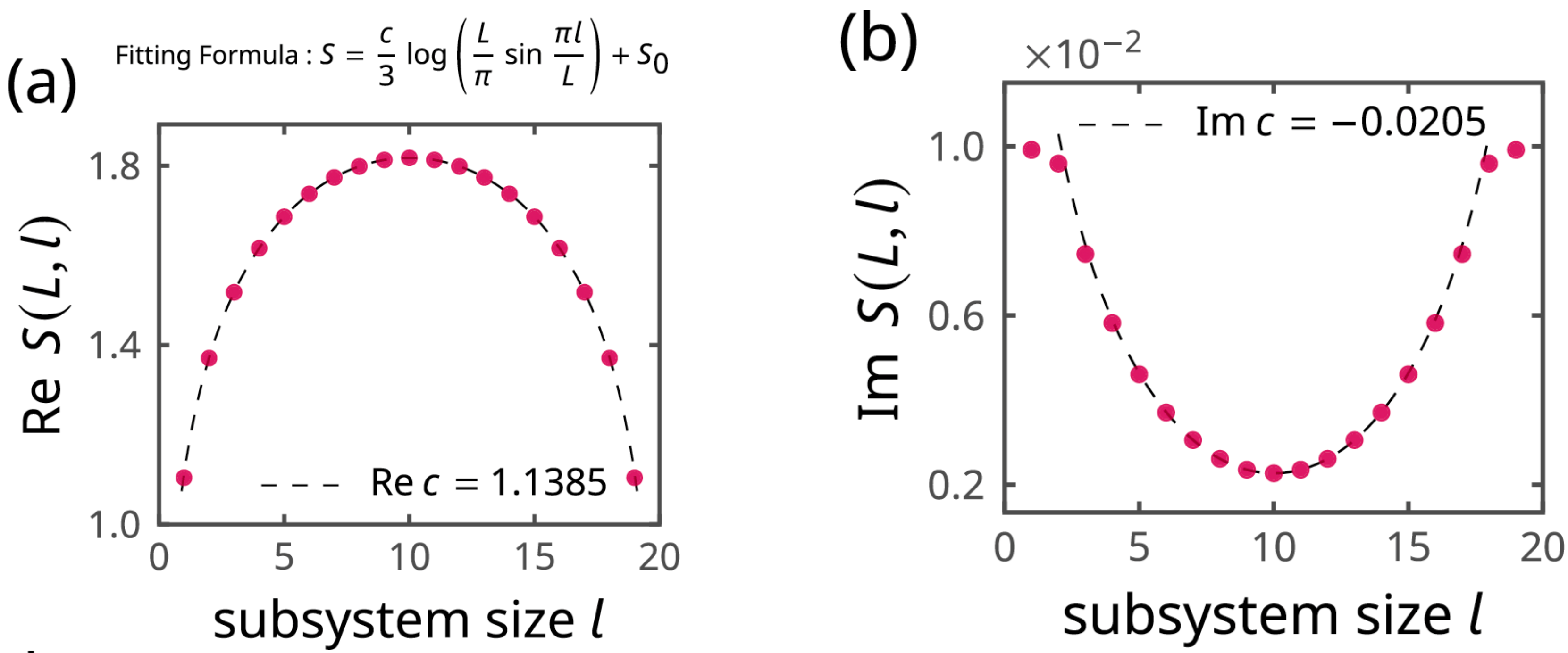
Numerical calculations up to L = 24 and D = 400 (bond dimension)

Potts model: $\hat{H}^{T} = \hat{H}$ $\longrightarrow$ $|\psi_{\mathrm{GS}}\rangle\rangle \propto |\psi_{\mathrm{GS}}\rangle^{*}$

Periodic boundary conditions

We generalize DMRG to non-Hermitian systems.

We numerically calculate the complex entanglement entropy and **confirm the CFT scaling**.



$$c \approx 1.1385 \pm 0.0205\mathrm{i} \quad (\mathrm{CFT}: c = 1.1376 \pm 0.0211\mathrm{i})$$

Numerical results

☆ **Numerical results of complex central charges**

	$c\,(D=300)$	$c\,(D=400)$	
CFT		$1.1376 - 0.0211\mathrm{i}$	analytical (CFT)
$L=4$	$1.1927 + 0.0104\mathrm{i}$	$1.1927 + 0.0104\mathrm{i}$	
$L=8$	$1.1429 - 0.0171\mathrm{i}$	$1.1429 - 0.0171\mathrm{i}$	
$L=12$	$1.1398 - 0.0196\mathrm{i}$	$1.1397 - 0.0195\mathrm{i}$	
$L=16$	$1.1389 - 0.0201\mathrm{i}$	$1.1388 - 0.0203\mathrm{i}$	
$L=20$	$1.1388 - 0.0209\mathrm{i}$	$1.1385 - 0.0205\mathrm{i}$	
$L=24$	$1.1391 - 0.0208\mathrm{i}$	$1.1384 - 0.0206\mathrm{i}$	numerical (DMRG)
$L=24\ (\hat{\rho}_{\mathrm{A}}^{\mathrm{R}})$	1.1635	1.1633	numerical (right GS)
$L=24\ (\hat{\rho}_{\mathrm{A}}^{\mathrm{SVD}})$	1.1283	1.1431	numerical (SVD EE)

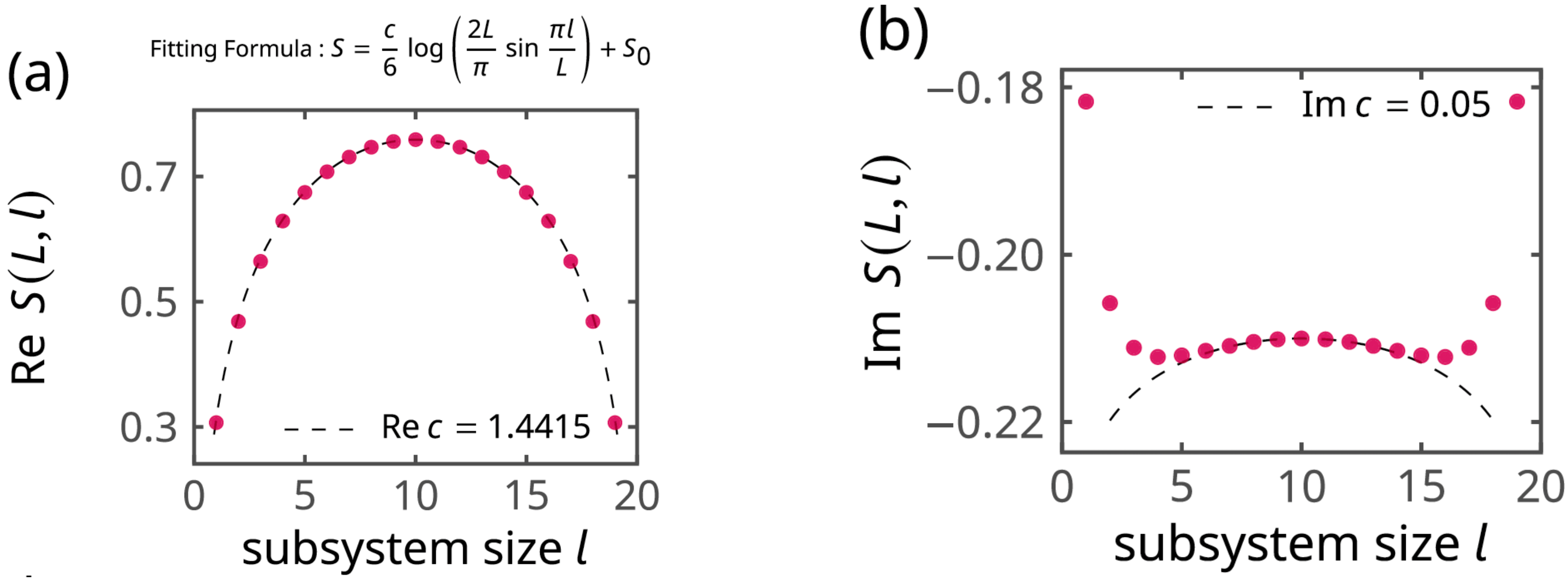
Parzygnat *et al.*, JHEP **2023**, 123 (2023)

☆ **Complex entanglement entropy characterizes nonunitary quantum critical phenomena!**

Open boundary conditions

Under OBC, the CFT scaling does not seem to hold.

Subtle boundary effects specific to non-Hermitian systems?



cf. Similar breakdown of the CFT scaling under OBC for c = -2

Chang *et al.*, Phys. Rev. Research **2**, 033069 (2020)

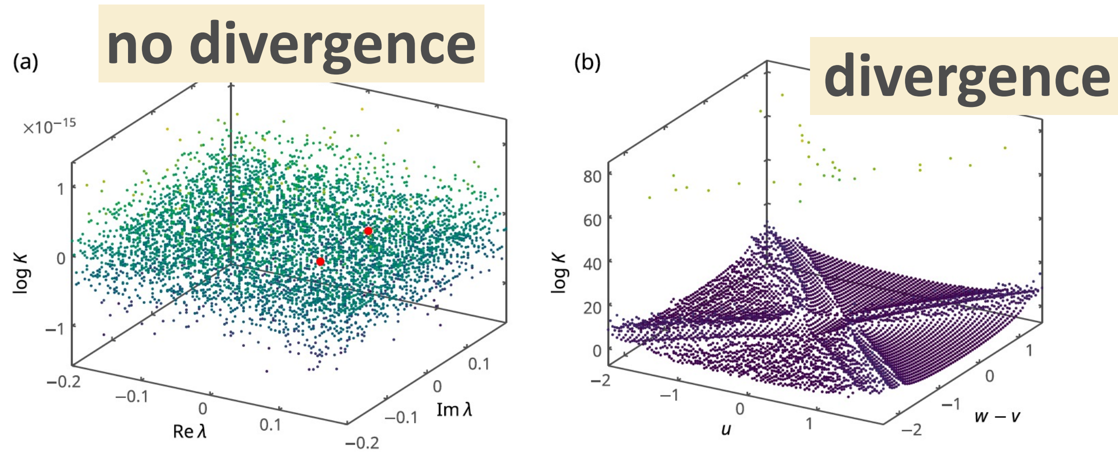
Distinction from real nonunitary CFT

Real nonunitary CFT: criticality accompanies an **exceptional point** (Hamiltonian is not diagonalizable)

divergence of $$K = \frac{\langle\psi|\psi\rangle \langle\langle\psi|\psi\rangle\rangle}{|\langle\langle\psi|\psi\rangle|^2}$$ (condition number, Petermann factor)

NH Potts

(c: complex)



NH SSH

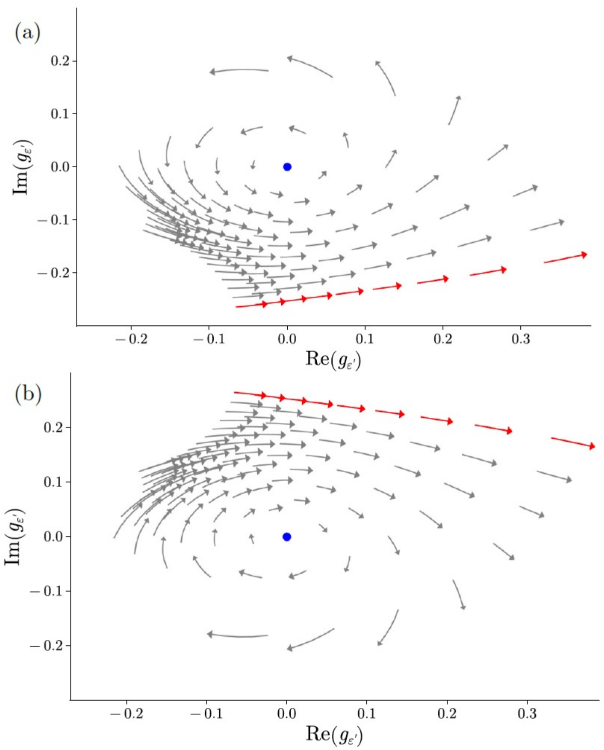
(c = -2)

☆ Distinct mechanisms of criticality between complex CFT and real nonunitary CFT!

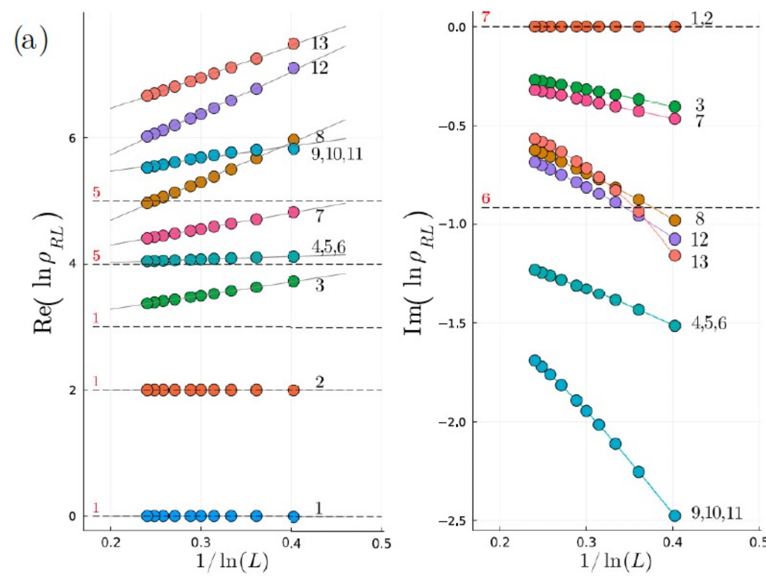
Further numerical results based on the tensor-network framework

Vander Linden *et al.*, PRB **113**, 205106 (2026)

spiral renormalization group flow



complex entanglement spectrum



Outline

1. Introduction

2. Complex conformal field theory (review)

3. Complex entanglement entropy for complex conformal field theory

4. Complex nonlinear sigma models

Motivation

So far, we have focused on Z_Q-symmetric Potts model.

Can we have complex fixed points with continuous symmetry?

How generic are complex fixed points?

As a prototype of models with continuous symmetry,
we study **complexified nonlinear sigma models:**

$$S = \frac{1}{2t} \int d^d \boldsymbol{r} \; \mathrm{tr}\left[(\nabla Q^\dagger)(\nabla Q)\right]$$

$$t \in \mathbb{R} \to t \in \mathbb{C}$$

Yamamoto & **Kawabata**,
PRB **114**, 115112 (2026)

Nonlinear sigma model

$$S = \frac{1}{2t} \int d^d \boldsymbol{r} \; \mathrm{tr} \left[(\nabla Q^\dagger)(\nabla Q) \right]$$

simplest symmetry-allowed gradient term

coupling constant t

N × *N* matrix field $Q \in \mathrm{O}\left(N\right), \mathrm{U}\left(N\right), \mathrm{Sp}\left(N\right), \cdots$

(tenfold target manifolds)

many applications in condensed matter physics:

– Anderson transitions in disordered electron systems

– topological insulators and superconductors

– Haldane gap in quantum antiferromagnetic chains

– monitored quantum dynamics and quantum error correction

Complex nonlinear sigma model

$$S = \frac{1}{2t} \int d^d \boldsymbol{r} \; \mathrm{tr}\left[(\nabla Q^\dagger)(\nabla Q) \right]$$

Usually, the coupling constant *t* is real.

→ **We complexify *t* and study complex nonlinear sigma models using perturbative renormalization group (RG).**

We show that complex fixed points arise generically.

In this talk, we focus on the O(*N*) model.

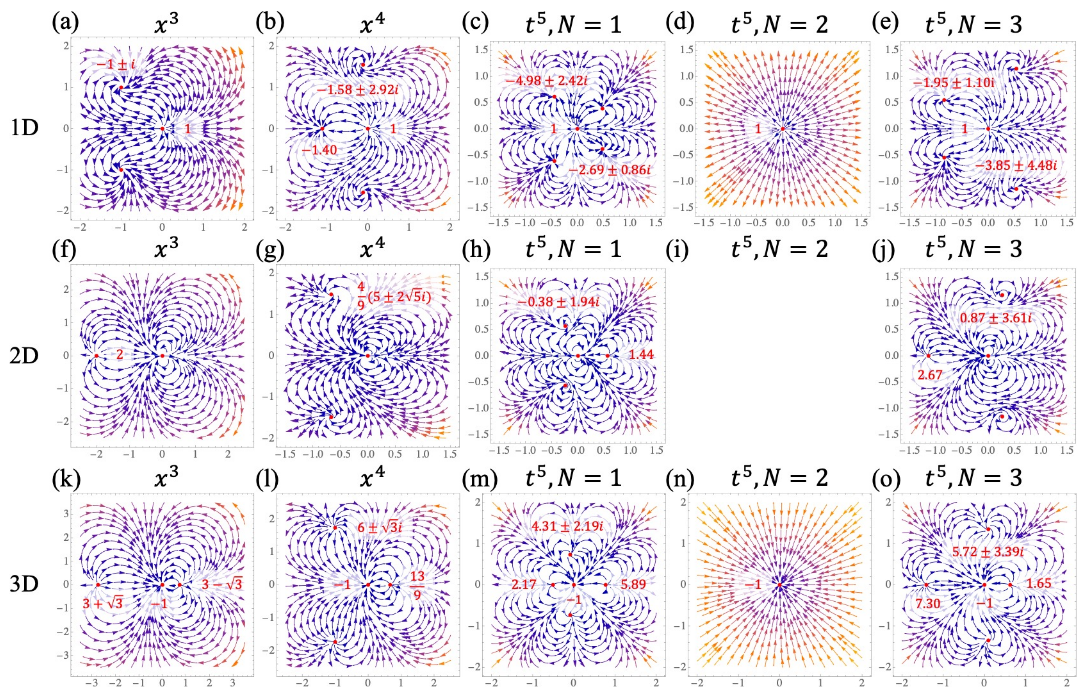
Complex renormalization group flow

Perturbative RG equation [O (t^5)]:

Hikami, Phys. Lett. B **98**, 208 (1981)
Wegner, Nucl. Phys. B **316**, 663 (1989)

$$\frac{dt}{dl} = \beta(t) = (2-d)\,t + (N-2)\,t^2 + \frac{1}{2}(N-2)^2 t^3 + \frac{3}{8}(N-2)^3 t^4 - (N-2)\,c_1(N)\,t^5$$

RG flow in the complex-t plane



Complex scaling dimensions (1)

e.g., $d = 3$, $N = 3$, O (t^4)

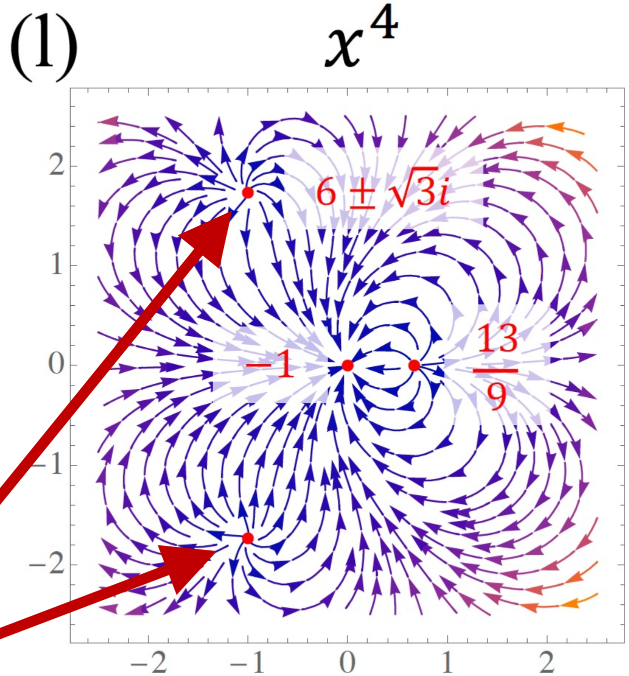
$$\frac{dt}{dl} = \beta(t) = -t + t^2 + \frac{1}{2}t^3 + \frac{3}{8}t^4$$

fixed points $\beta(t_c) = 0$

$$\longrightarrow \quad t_c = 0, \frac{2}{3}, -1 \pm \sqrt{3}\mathrm{i}$$

real fixed points (conventional)

complex fixed points



The corresponding scaling dimensions are also complex:

$$y_t = \beta'(t_c) = -1, \frac{13}{9}, \underline{6 \mp \sqrt{3}\mathrm{i}}$$

Complex scaling dimensions (2)

$$t - t_{\mathrm{c}} \propto e^{y_t l} \quad (y_t = 6 \mp \sqrt{3}\mathrm{i})$$

unstable spiral of renormalization-group flow

(unconventional in unitary theory)

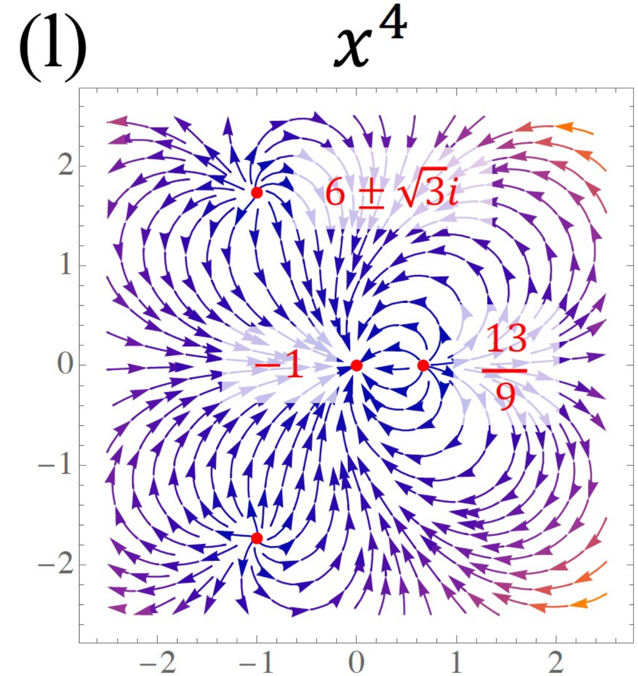
$\mathrm{Re}\, y_t$: stability of the fixed point

(similar to unitary theory)

$\mathrm{Im}\, y_t$: angular velocity of the spiral flow

critical behavior of the correlation length:

$$\xi \propto \left| t - t_{\mathrm{c}} \right|^{-1/y_t}$$



2D: complex critical phenomena (1)

In the conventional O($N > 2$) model with the real coupling, we do not have nontrivial fixed points.

(cf. asymptotic freedom)

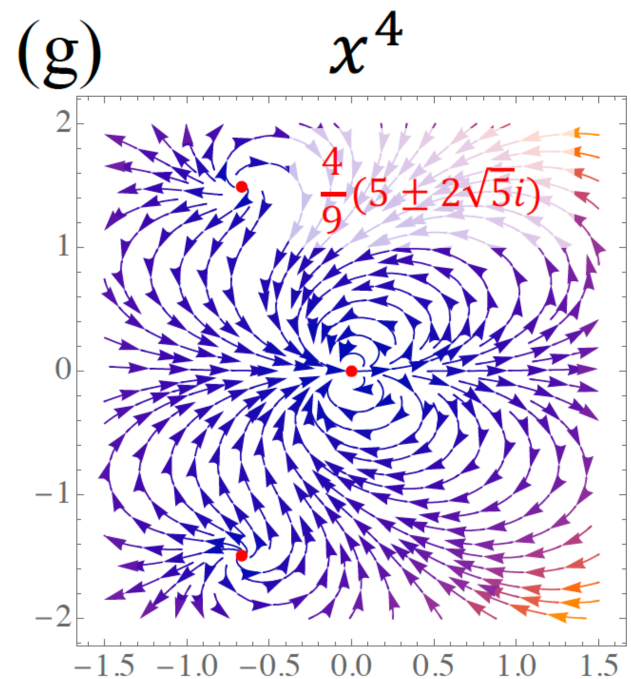
By contrast, O($N > 2$) model can support nontrivial complex fixed points!

[$N = 3$, O (t^4)]

$$t_c = \frac{2}{3}\left(-1 \pm \sqrt{5}\mathrm{i}\right)$$

$$y_t = \frac{4}{9}\left(5 \mp 2\sqrt{5}\mathrm{i}\right)$$

$$\approx 2.22222 \mp 1.98762\mathrm{i}$$



2D: complex critical phenomena (2)

Nonperturbative CFT analysis:

Yang & Scaffidi, PRL **137**, 051601 (2026)

(Coulomb-gas method for O(*N*) loop models)

$$\nu = \frac{1}{y_t} = \frac{1}{4}\left(1 \pm \frac{\mathrm{i}\pi}{\cosh^{-1}(N/2)}\right)$$

$$\nu = \frac{1}{y_t} \approx \begin{cases} 0.25 \pm 0.223607\mathrm{i} & \text{(perturbative)} \\ 0.25 \pm 0.816063\mathrm{i} & \text{(nonperturbative)} \end{cases} \qquad (N=3)$$

The presence of complex fixed points is confirmed nonperturbatively.

Complex scaling dimensions quantitatively differ, but are not so different.

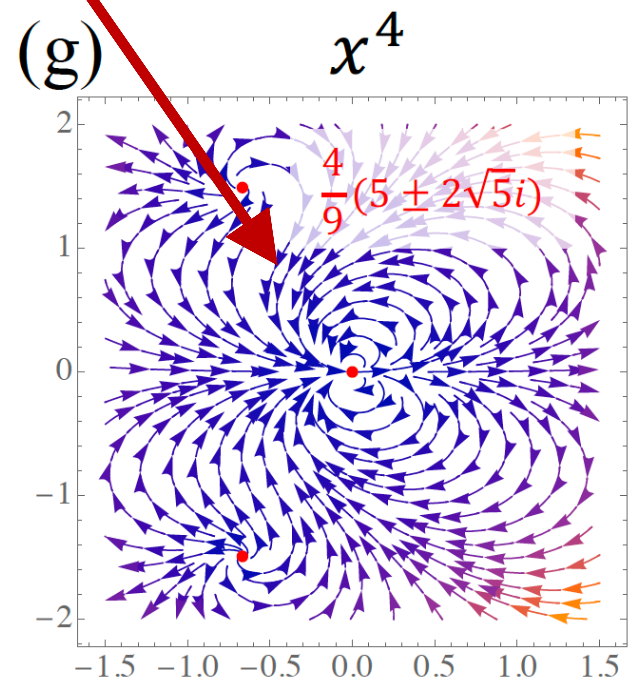
Lattice realization: non-Hermitian *S*=1 quantum antiferromagnet

$$\hat{H} = \sum_n \left[J_1 \hat{\boldsymbol{S}}_n \cdot \hat{\boldsymbol{S}}_{n+1} + J_2 \hat{\boldsymbol{S}}_n \cdot \hat{\boldsymbol{S}}_{n+2} + K \left(\hat{\boldsymbol{S}}_n \cdot \hat{\boldsymbol{S}}_{n+1} \right)^2 \right]$$

critical point: $J_1 = 1, \quad J_2 = 0.0660 \pm 0.338\mathrm{i}, \quad K = 0.176 \pm 0.335\mathrm{i}$

Another 2D feature: circular renormalization group flow

A flow that initially departs from weak coupling $t = 0$ eventually returns to it.



Generic feature of

$$\frac{dt}{dl} \simeq (N-2)\, t^2 \qquad (t \in \mathbb{C})$$

(The linear term is absent in the beta function)

cf. Non-Hermitian Kondo model

$$\frac{d\Gamma}{d\log D} = -\rho\Gamma^2 + n\rho^2\Gamma^3 + c\rho^3\Gamma^4 + \cdots \qquad (\Gamma \in \mathbb{C})$$

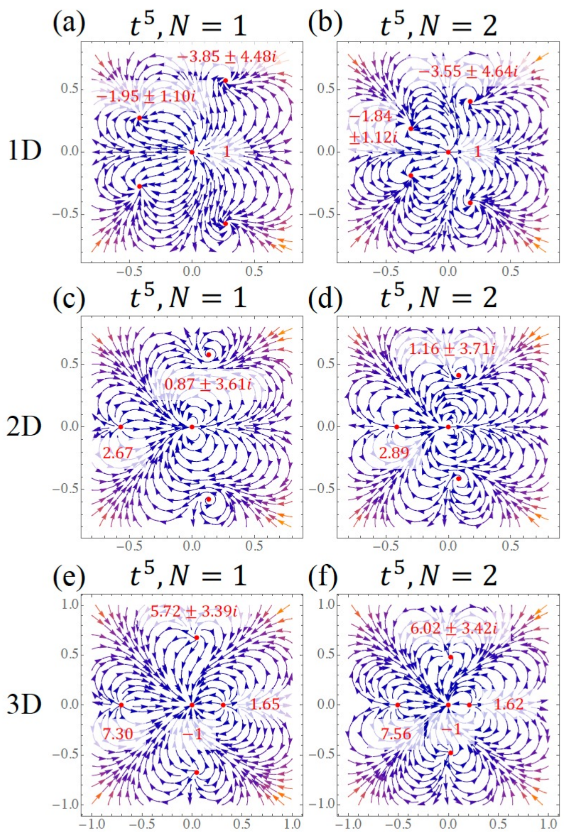
Nakagawa *et al.*, PRL **121**, 203001 (2018)

Tenfold symmetric spaces

Extended to the other nine symmetric spaces

$$Q \in \mathrm{U}\left(N\right), \mathrm{Sp}\left(N\right), \mathrm{U}\left(N\right)/\mathrm{O}\left(N\right), \cdots$$

e.g., Sp(N)



Qualitatively similar RG flows

Different complex scaling dimensions (critical exponents)

Complex fixed points generally appear in nonlinear sigma models.

Summary

- Complex CFT describes unique critical phenomena that have no counterparts in Hermitian systems.

- We show that complex entanglement entropy captures complex CFT.

- We study complex nonlinear sigma models as a prototype of complex CFT.