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THE INSTITUTE FOR SOLID STATE PHYSICS
THE UNIVERSITY OF TOKYO

Non-Hermitian Topology in Hermitian Topological Matter: Wannier Localizability and Detachable Topological Boundary States

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PRX Quantum 4, 030315 (2023)
arXiv:2405.10015, 2407.09458, 2407.18273

PRX Quantum 4, 030315 (2023)

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arXiv:2405.10015

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arXiv:2407.09458 & 2407.18273

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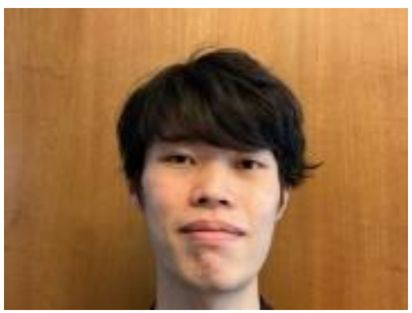

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Outline

1. Introduction

2. Non-Hermitian topology (review)

3. Hermitian bulk – non-Hermitian boundary correspondence

4. Non-Hermitian topology in Hermitian topological matter

5. Non-Hermitian origin of Wannier localizability and detachable topological boundary states

Non-Hermitian physics

Despite the enormous success, the existing framework of topological phases is **confined to Hermitian systems at equilibrium**.

→ **Richer properties appear in non-Hermitian systems!**

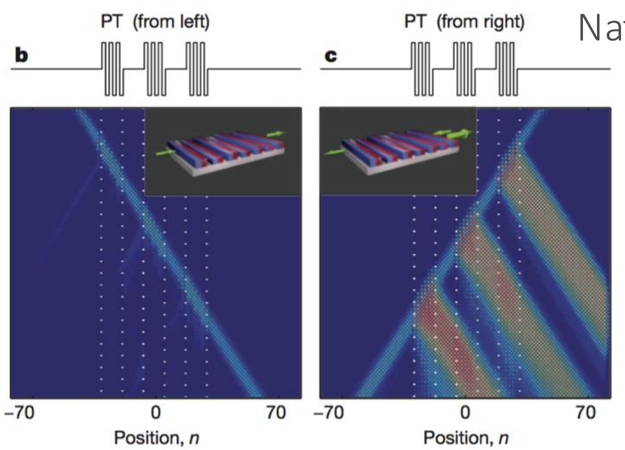
☆ Non-Hermiticity arises from **dissipation**, i.e., exchanges of energy or particles with an environment.

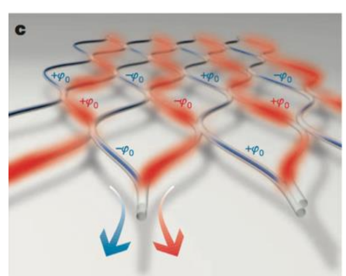
El-Ganainy *et al.*, Nat. Phys. **14**, 11 (2018)

- **Photonic lattices with gain/loss**

Unidirectional light transport

Regensburger *et al.*, Nature **488**, 167 (2012)

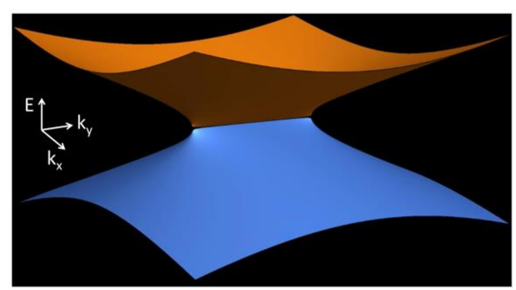


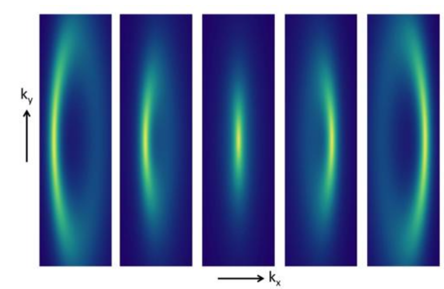


- **Finite-lifetime quasiparticles**

Bulk Fermi arc due to non-Hermitian self-energy

Kozii & Fu, arXiv: 1708.05841

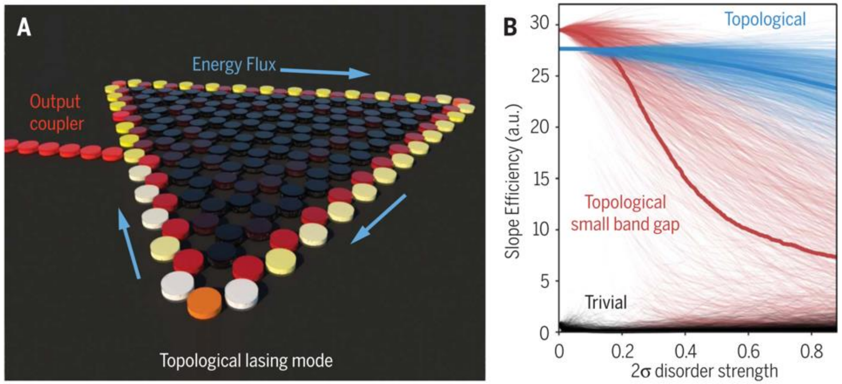




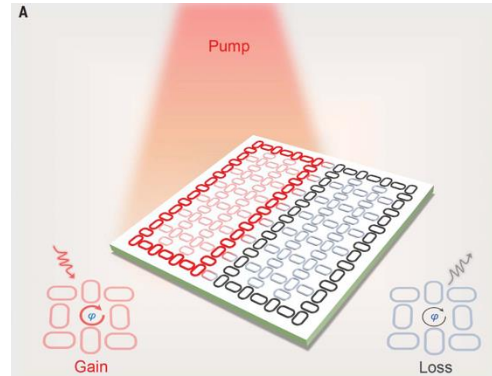
Non-Hermitian topological systems

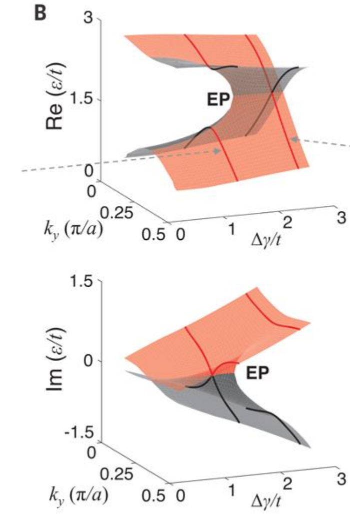
- **Topological laser**

New laser with high efficiency due to the **interplay of non-Hermiticity and topology**.



Bandres *et al.*, Science **359**, eaar4005 (2018)

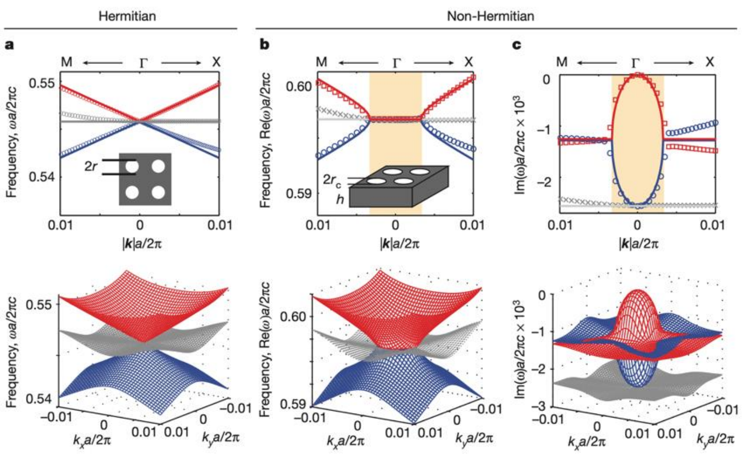




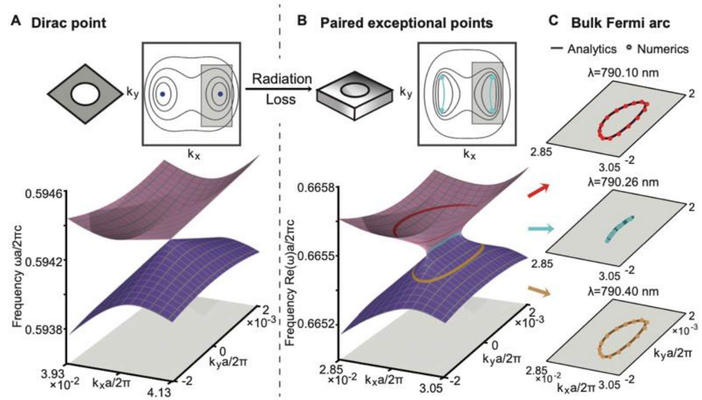
Zhao *et al.*, Science **365**, 1163 (2019)

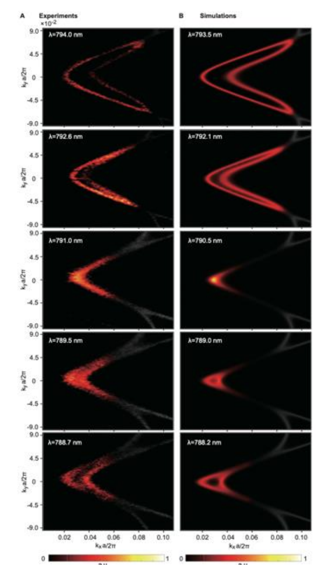
- **Exceptional point**

Nondiagonalizable gapless point (Jordan matrix)



Zhen *et al.*, Nature **525**, 354 (2015)





Zhou *et al.*, Science **359**, 1009 (2018)

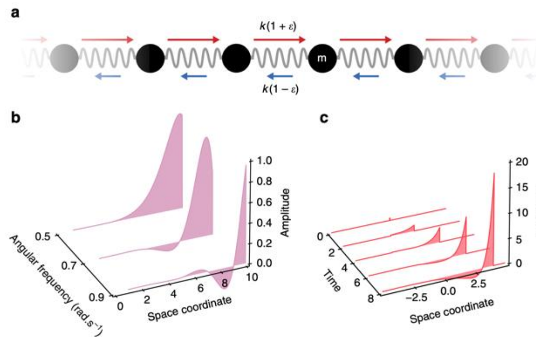
Non-Hermitian skin effect

☆ Non-Hermitian skin effect

Lee, PRL **116**, 133903 (2016); Yao & Wang, PRL **121**, 086803 (2018); Kunst *et al.*, PRL **121**, 026808 (2018)

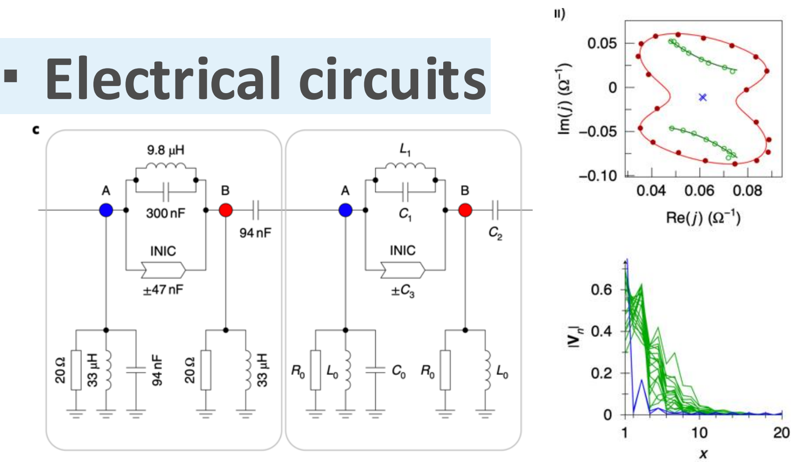
Localization of an extensive number of eigenmodes due to non-Hermitian topology

▪ Mechanical metamaterials



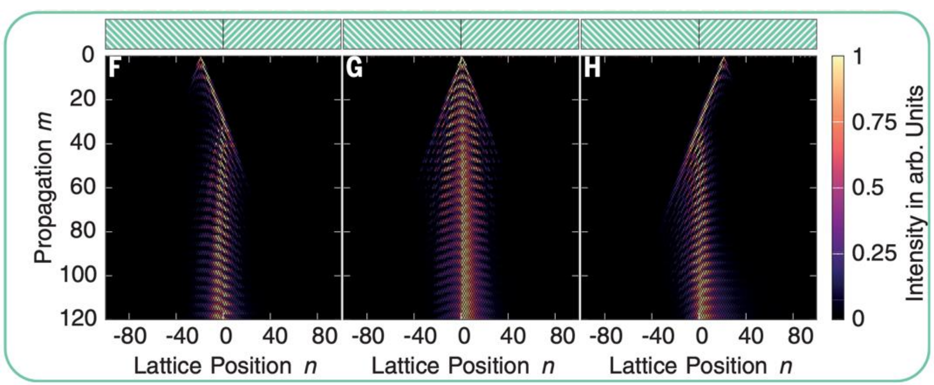
Brandenbourger *et al.*, Nat. Commun. **10**, 4608 (2019)

▪ Electrical circuits



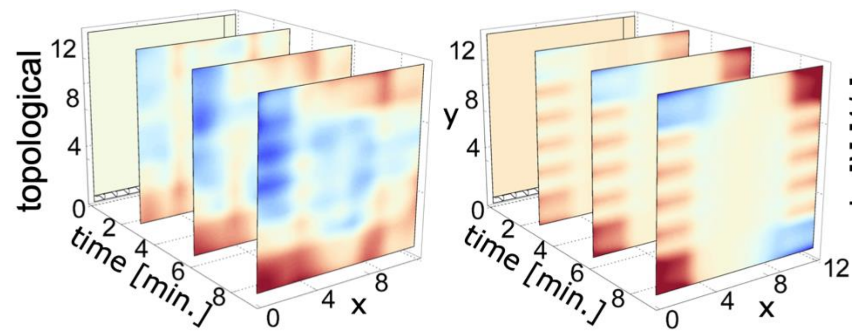
Helbig *et al.*, Nat. Phys. **16**, 747 (2020)

▪ Photonic lattice



Weidemann *et al.*, Science **368**, 311 (2020)

▪ Active matter



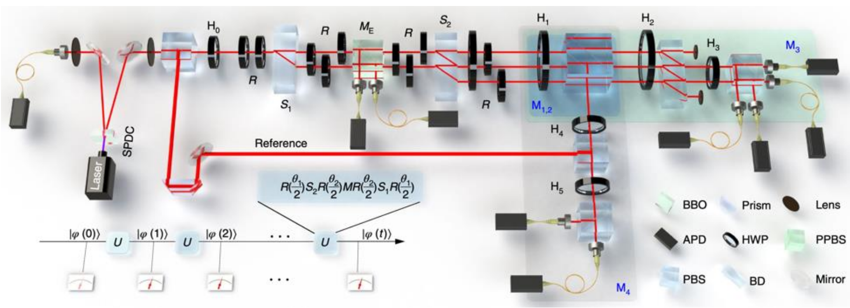
Palacios *et al.*, Nat. Commun. **12**, 4691 (2021)

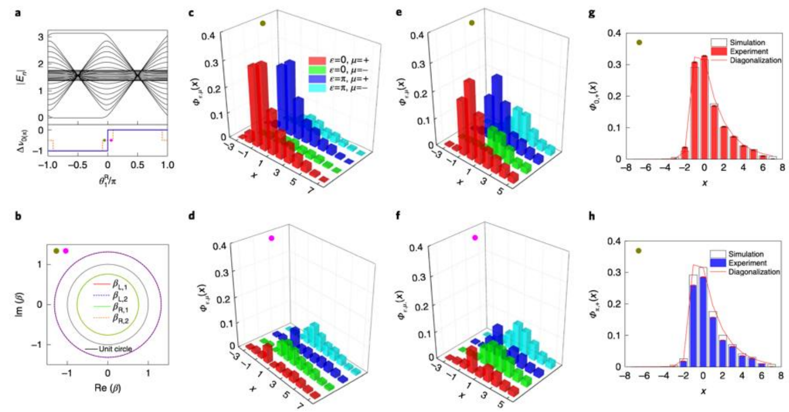
Skin effect in quantum physics

Skin effect has been observed also in recent quantum experiments.

- **Quantum walk (single photons)**

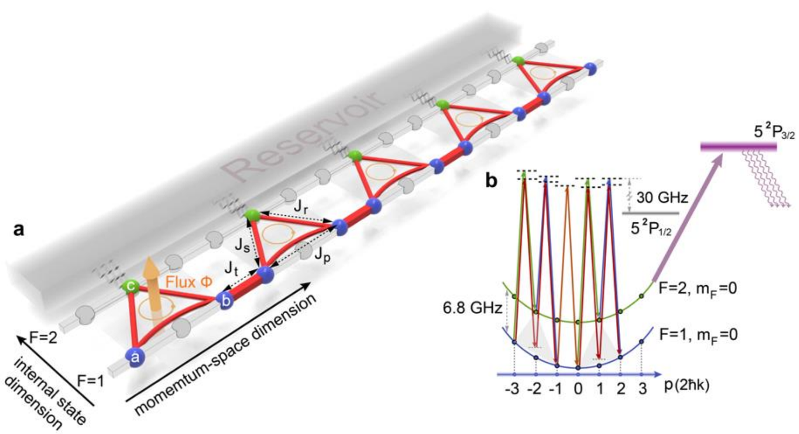
Xiao *et al.*, Nat. Phys. **16**, 761 (2020)

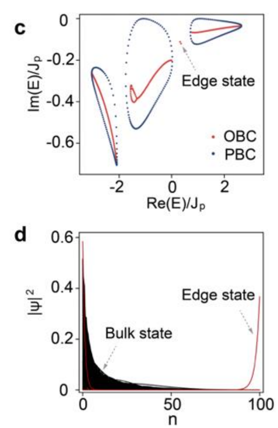


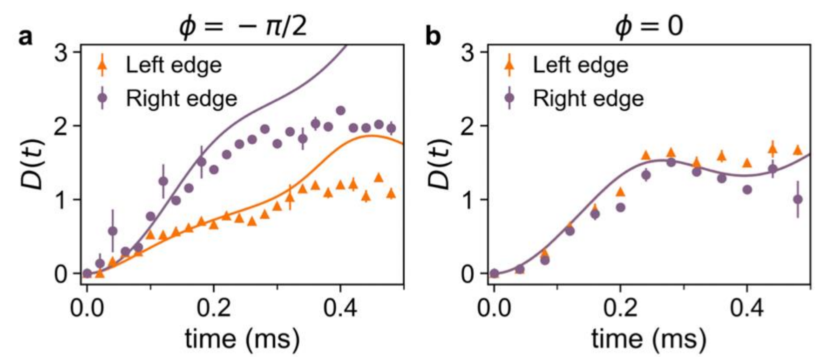


- **Ultracold atoms**

Liang *et al.*, PRL **129**, 070401 (2022)







Skin effect in quantum physics

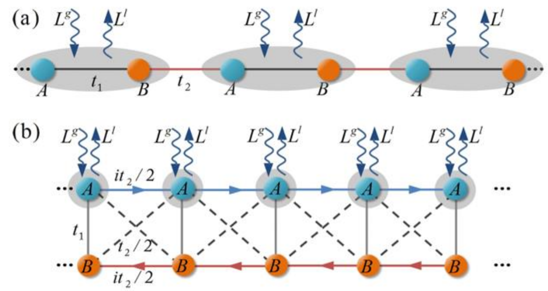
Non-Hermitian topology is relevant even to more generic open quantum systems that are not characterized by Hamiltonians.

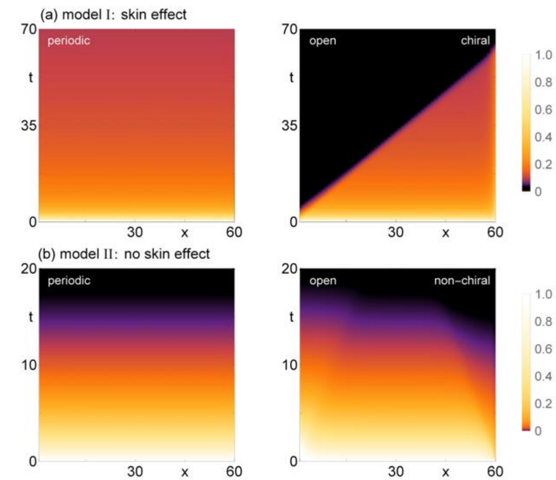
Song, Yao & Wang, PRL **123**, 170401 (2019)

- **Master equation**

$$d\hat{\rho}/dt = \underline{\mathcal{L}}\hat{\rho}$$

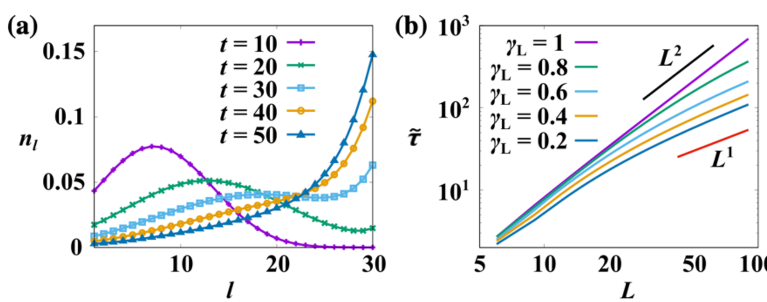
Non-Hermitian superoperator (Liouvillian)

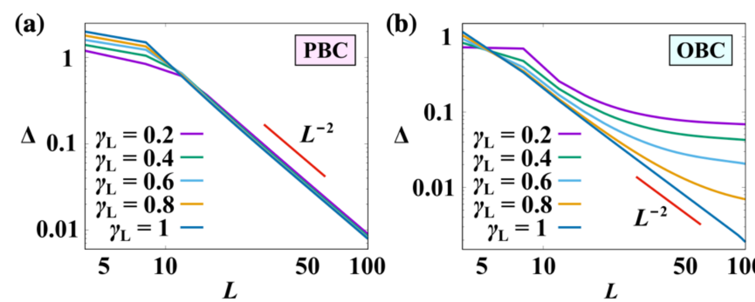




Chiral damping due to the skin effect

Slowdown of relaxation due to the skin effect





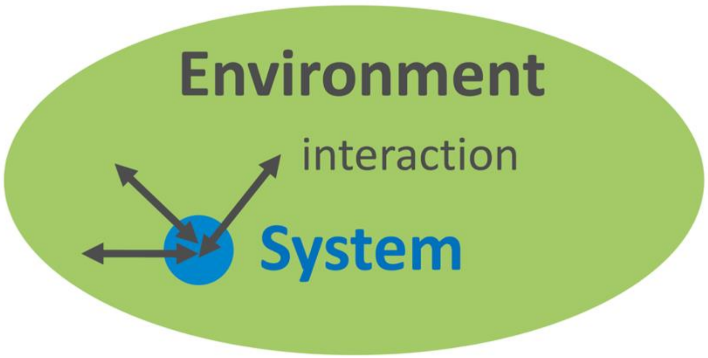
Haga *et al.*, PRL **127**, 070402 (2021); Mori *et al.*, PRL **125**, 230604 (2020)

Motivation

Non-Hermiticity gives rise to unique topological phases that have no Hermitian counterparts.

Meanwhile, open systems can be embedded in larger closed systems.

→ **Can we connect intrinsic non-Hermitian topology with Hermitian topology?**



Results (1)

We find a new topological correspondence between Hermitian bulk and non-Hermitian boundary.

***d*-dim Hermitian bulk** ············ **(*d*-1)-dim non-Hermitian boundary**

Topological boundary modes + non-Hermiticity (dissipation)

→ unique non-Hermitian topology

New boundary phenomena:

– corner skin effect of chiral edge modes

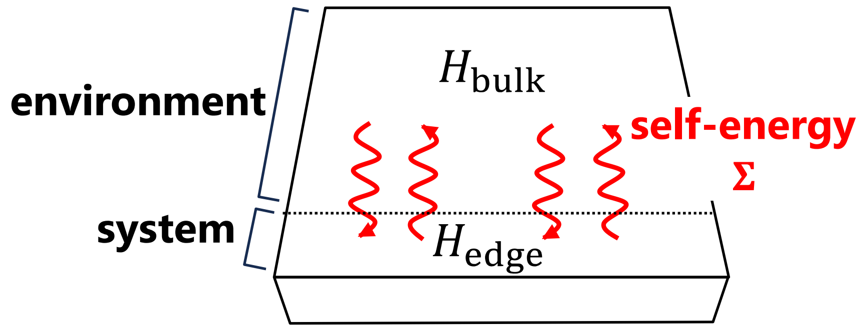
– chiral hinge modes due to non-Hermiticity

Schindler, Gu, Lian & **Kawabata**, PRX Quantum **4**, 030315 (2023)

cf. Nakamura, Inaka, Okuma & Sato, PRL **131**, 256602 (2023)

Results (2)

We develop the same correspondence even in Hermitian topological insulators (without non-Hermicity)



The self-energy captures the particle exchange between the bulk and boundary, and detects Hermitian topology in the bulk and non-Hermitian topology at the boundary.

Hamanaka, Yoshida & **Kawabata**, arXiv:2405.10015

Results (3)

Our correspondence is also helpful in understanding Hermitian physics

We develop the classification of Wannier localizability and detachable boundary states in Hermitian topological insulators

– Intrinsic NH topology: Wannier obstructions

– Extrinsic NH topology: **NO** Wannier obstructions

detachable topological boundary states

Nakamura, Shiozaki, Shimomura, Sato & **Kawabata**, arXiv:2407.09458 & 2407.18273

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Complex-energy gaps (1)

An “**energy gap**” is needed to define a topological phase.

However, a non-Hermitian extension of an **“energy gap”** is nontrivial since the **spectrum is complex**.

Hermitian

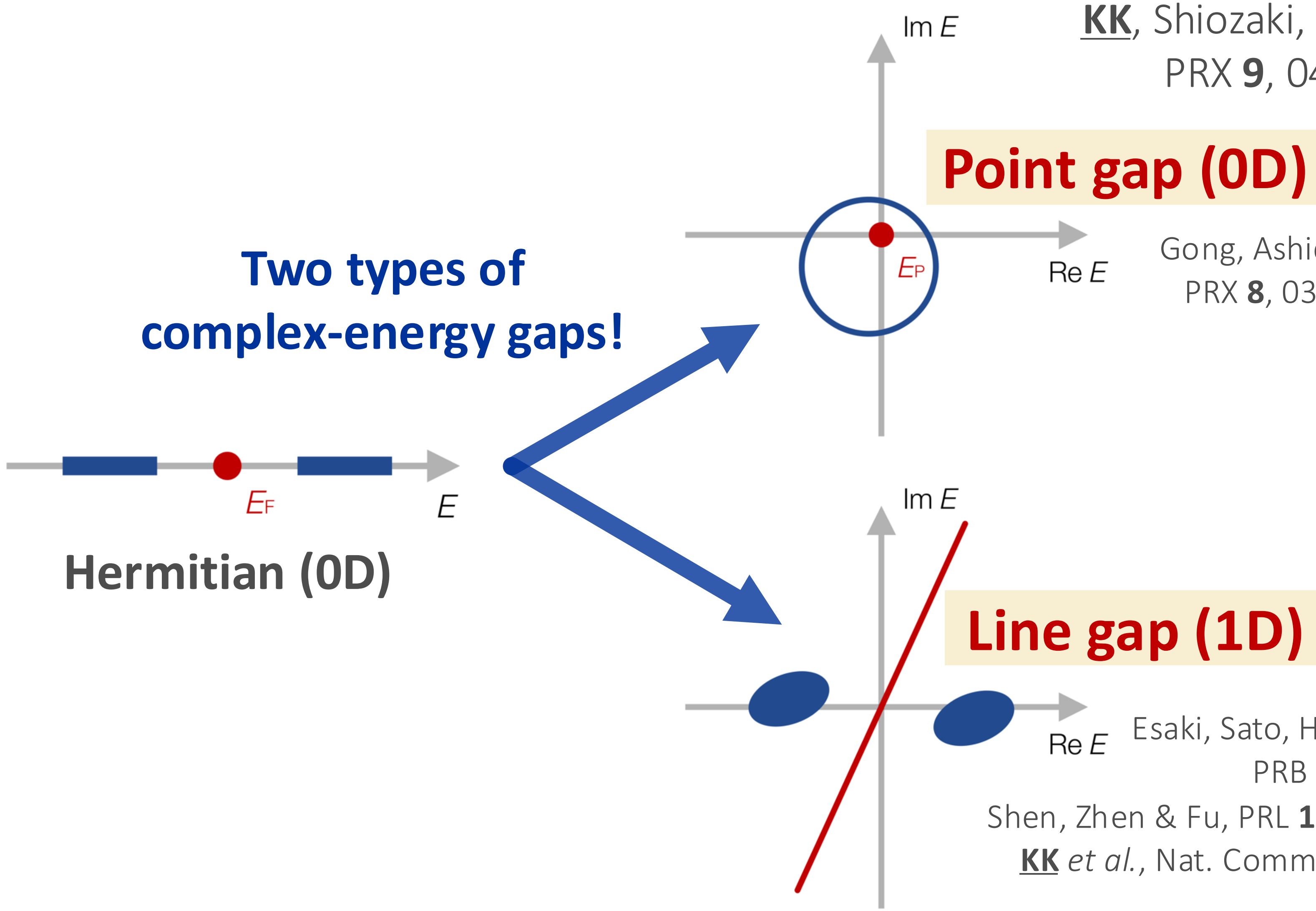
Energy gap in Hermitian systems:

- Energy regions where states are **forbidden to be present**.
- They **should be point-like (0D)** since the **real spectrum is 1D**.

→ Since the **complex spectrum is 2D (real and imaginary)**, such vacant regions can be **either 0D or 1D**!

Complex-energy gaps (2)

KK, Shiozaki, Ueda & Sato, PRX **9**, 041015 (2019)



Gong, Ashida, **KK** *et al.*, PRX **8**, 031079 (2018)

Esaki, Sato, Hasebe & Kohmoto, PRB **84**, 205128 (2011)
Shen, Zhen & Fu, PRL **120**, 146402 (2018)
KK *et al.*, Nat. Commun. **10**, 297 (2019)

NOTE: The definition that should be adopted depends on the individual physical situations that we are interested in.

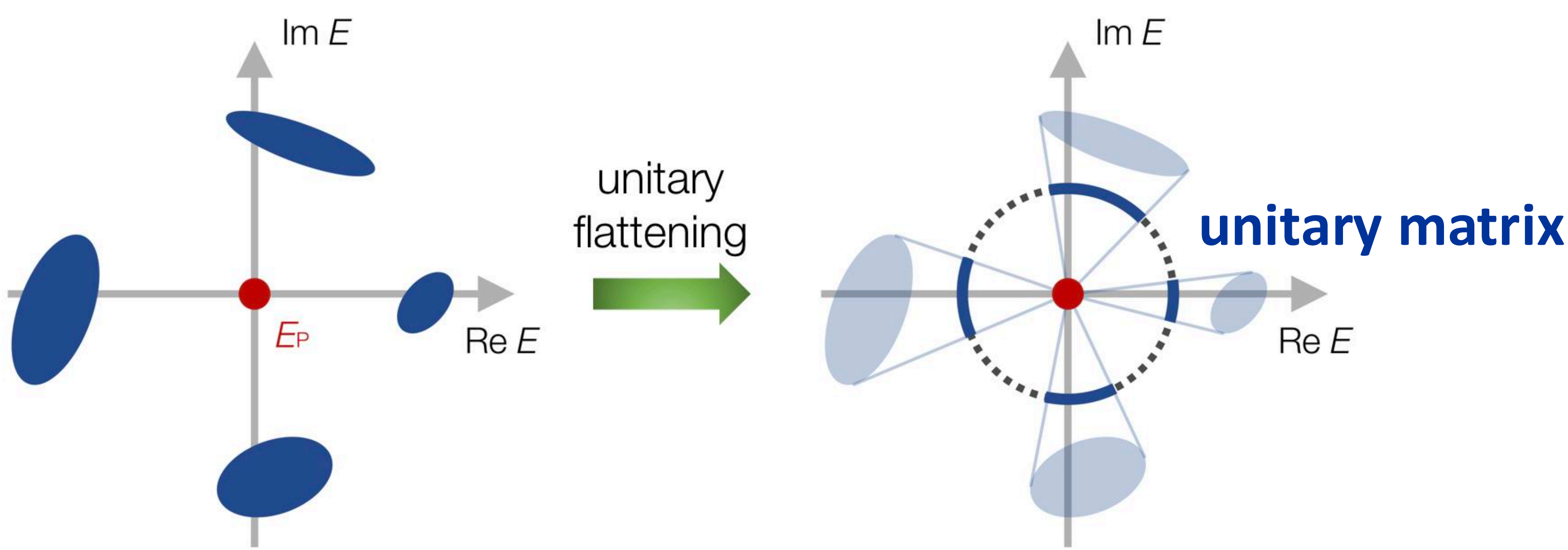
Point gap: unitary flattening

Unitary flattening for a point gap:

A non-Hermitian Hamiltonian with a **point gap** can be continuously deformed into a **unitary matrix** while having symmetry and the point gap.

non-Hermitian H $\longleftrightarrow$ unitary U $\longleftrightarrow$ **Hermitian** $\begin{pmatrix} 0 & U \\ U^\dagger & 0 \end{pmatrix}$

Classification is well established!



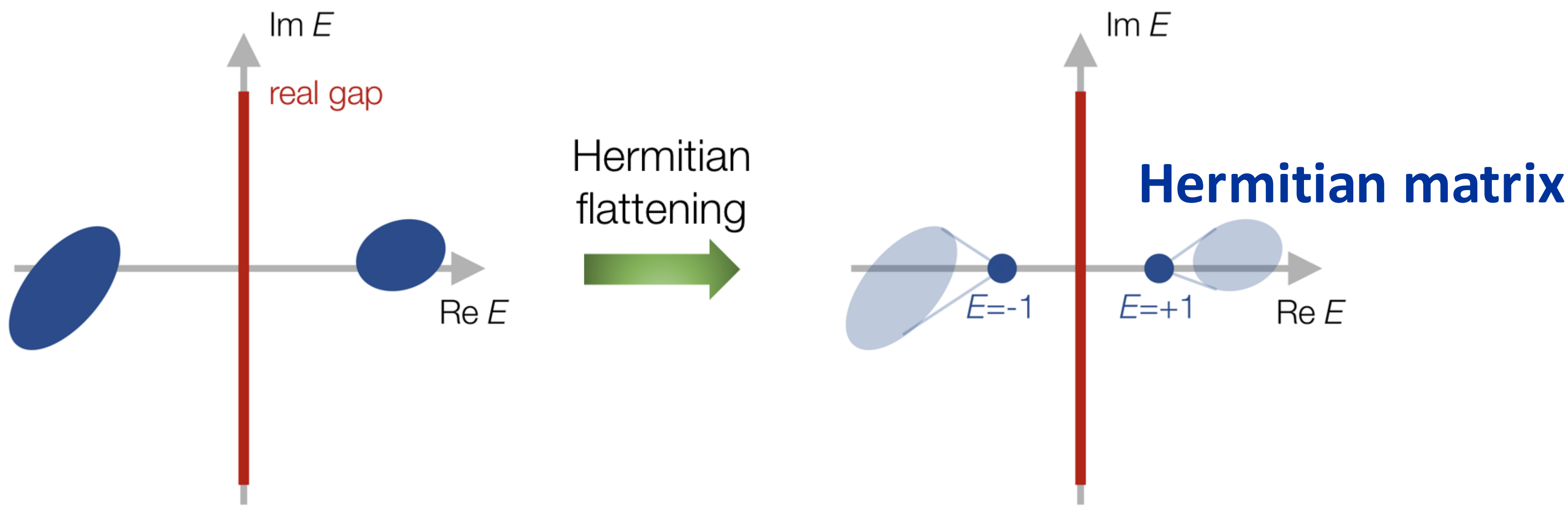
Line gap: Hermitian flattening

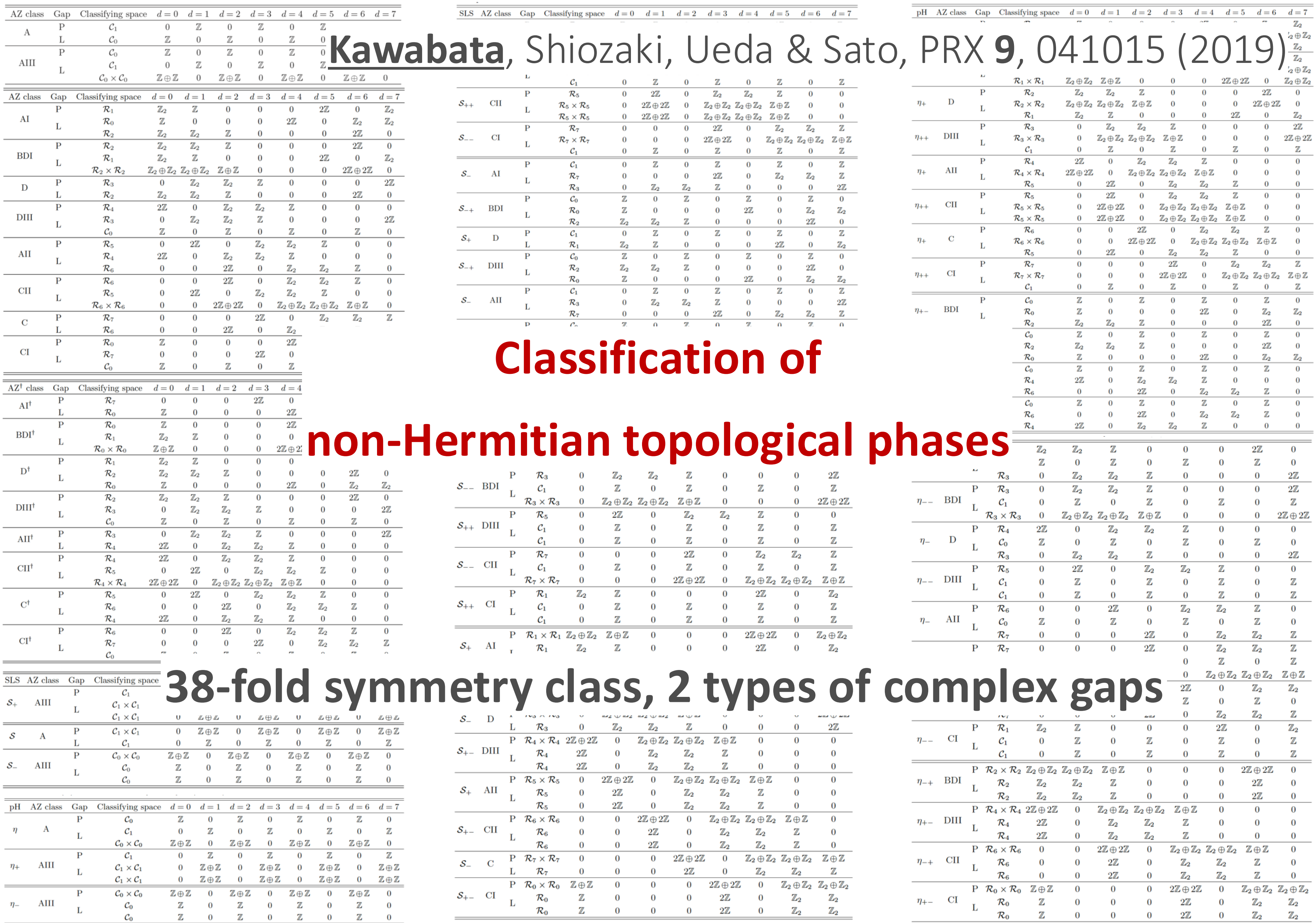
Hermitian flattening for a line gap:

A non-Hermitian Hamiltonian with a **line gap** can be continuously deformed into a **Hermitian matrix** while having symmetry and the line gap.

non-Hermitian H ⟷ **Hermitian** $\tilde{H}$

Classification is well established!





Kawabata, Shiozaki, Ueda & Sato, PRX **9**, 041015 (2019)

Classification of non-Hermitian topological phases

38-fold symmetry class, 2 types of complex gaps

Intrinsic non-Hermitian topology

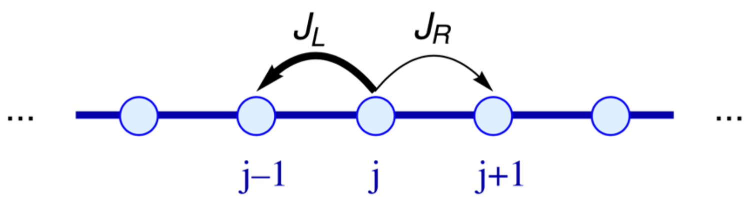
Line-gap topology: stability of Hermitian topology against non-Hermiticity

⟷ **Point-gap topology: intrinsic non-Hermitian topology**

▪ Hatano-Nelson model

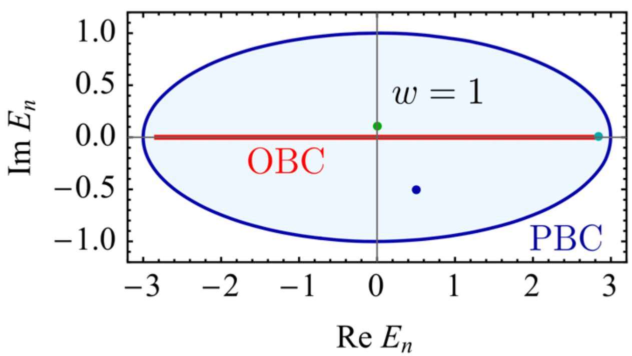
Hatano & Nelson, PRL **77**, 570 (1996)

asymmetric hopping



$$\hat{H}_{\mathrm{HN}} = \sum_i \left(J_{\mathrm{R}} \hat{c}_{i+1}^\dagger \hat{c}_i + J_{\mathrm{L}} \hat{c}_i^\dagger \hat{c}_{i+1} \right)$$

$$H_{\mathrm{HN}}(k) = J_{\mathrm{R}} e^{\mathrm{i}k} + J_{\mathrm{L}} e^{-\mathrm{i}k}$$



Winding of complex energy!

$$W(E) := \oint \frac{dk}{2\pi \mathrm{i}} \frac{d}{dk} \log \det \left(H(k) - E \right)$$

(ill-defined in Hermitian systems)

Gong, Ashida, **KK** *et al.*, PRX **8**, 031079 (2018)

☆ Skin effect: bulk-boundary correspondence for point-gap topology

Emergence of an extensive number of boundary modes!

Okuma, **KK**, Shiozaki & Sato, PRL **124**, 086801 (2020); Zhang, Yang & Fang, PRL **125**, 126402 (2020)

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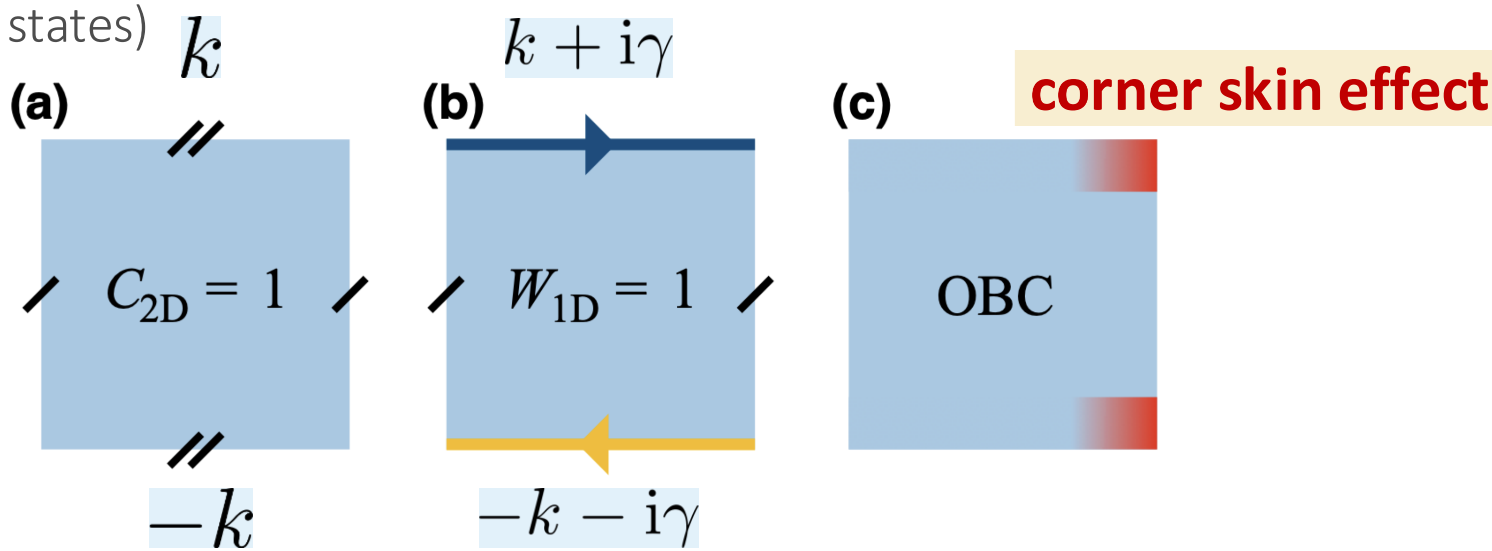
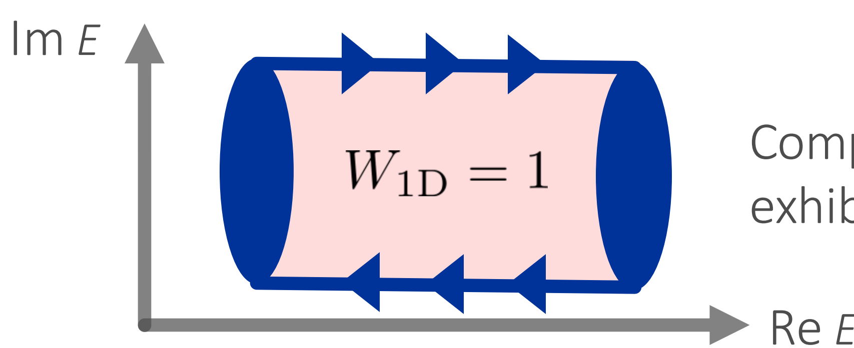
Non-Hermitian chiral edge modes

☆ Chiral edge modes in 2D topological insulators exhibit 1D non-Hermitian topology and skin effect!

(chiral edge states)



(add gain/loss at boundaries)

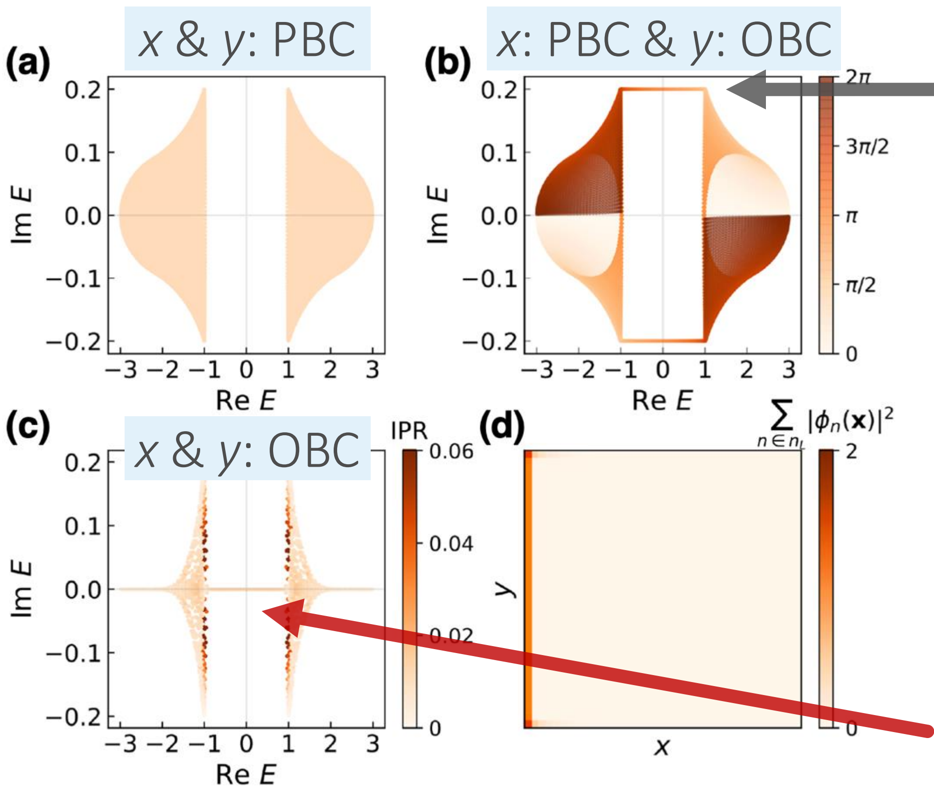


Complex chiral edge modes exhibit the spectral winding

Non-Hermitian Chern insulator

cf. **Kawabata** *et al.*, PRB **98**, 165148 (2018)

$$H(\boldsymbol{k}) = (m + t\cos k_x + t\cos k_y)\,\sigma_x + (t\sin k_x + \mathrm{i}\gamma)\,\sigma_y + (t\sin k_y)\,\sigma_z$$



chiral edge states

$\sin k_x + \mathrm{i}\gamma$

$$C_{\mathrm{2D}} = W_{\mathrm{1D}}$$

(can be shown in the presence of pseudo inversion symmetry)

New type of corner skin effect

– We can realize boundary non-Hermitian topology by a non-Hermitian perturbation only at the boundary.

cf. Nakamura, Inaka, Okuma & Sato, PRL **131**, 256602 (2023)

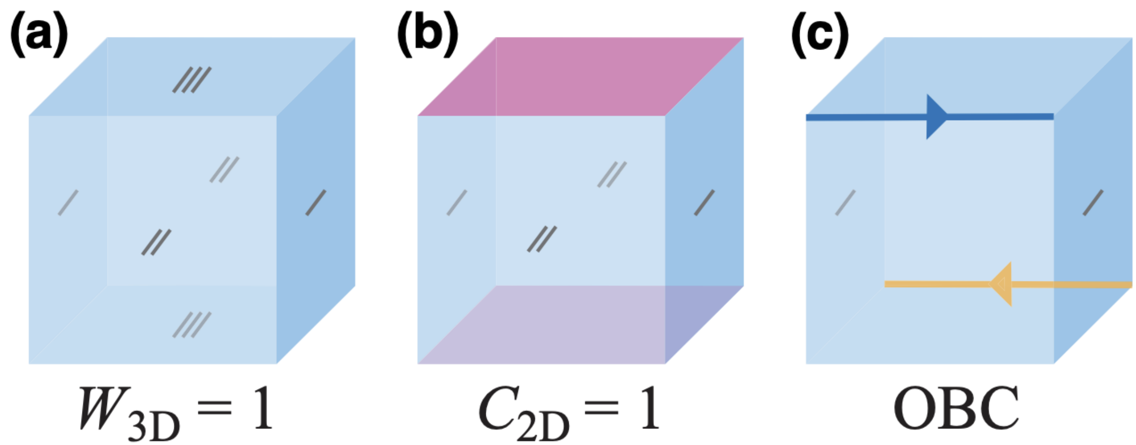
Non-Hermitian Dirac surface modes

☆ **Dirac surface states in 3D topological insulators exhibit 2D non-Hermitian topology and chiral hinge states!**

(non-Hermitian Dirac surface states)

$$k_x \sigma_x + k_y \sigma_y + \mathrm{i}\gamma k_x$$

2D point-gap topology



chiral hinge states $\mathrm{i}\gamma k_x$

$$W_{3\mathrm{D}} = C_{2\mathrm{D}}$$

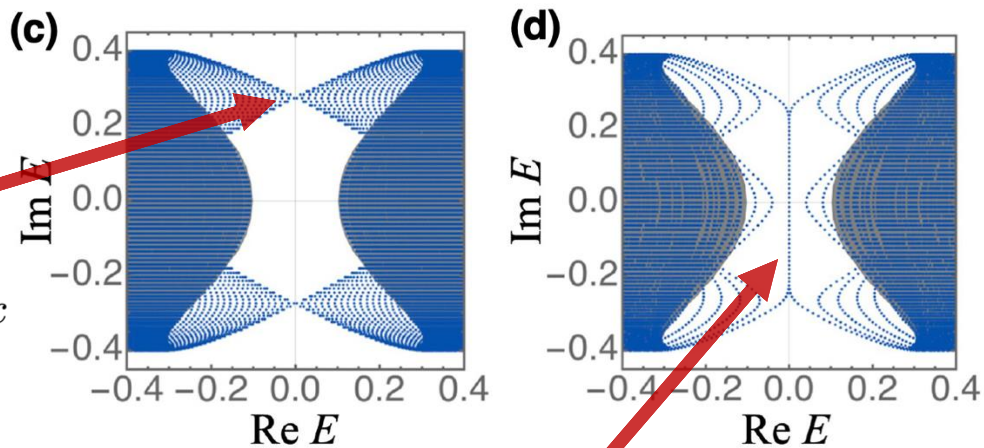
(3D Hermitian bulk) **(2D non-Hermitian boundary)**

Non-Hermitian 3D topological insulator

$$H(\boldsymbol{k}) = (m + t_1 \cos k_x + t_1 \cos k_y + t_1 \cos k_z)\,\tau_y + t_2\,(\sigma_x \sin k_x + \sigma_y \sin k_y + \sigma_z \sin k_z)\,\tau_x + \delta\,(\cos k_x + \cos k_y)\,\sigma_y \tau_y + \underline{\mathrm{i}\gamma \sin k_x}$$

x & *y*: PBC, *z*: OBC

x: PBC, *y* & *z*: OBC



Dirac surface states

$$k_x\sigma_x + k_y\sigma_y + \mathrm{i}\gamma k_x$$

chiral hinge states! $\mathrm{i}\gamma k_x$

New type of second-order topological insulator induced by non-Hermiticity!

Classification

☆ We classify possible non-Hermitian boundary topology for all the tenfold classes of topological insulators.

Different non-Hermitian boundary phenomena appear for different symmetry classes and spatial dimensions.

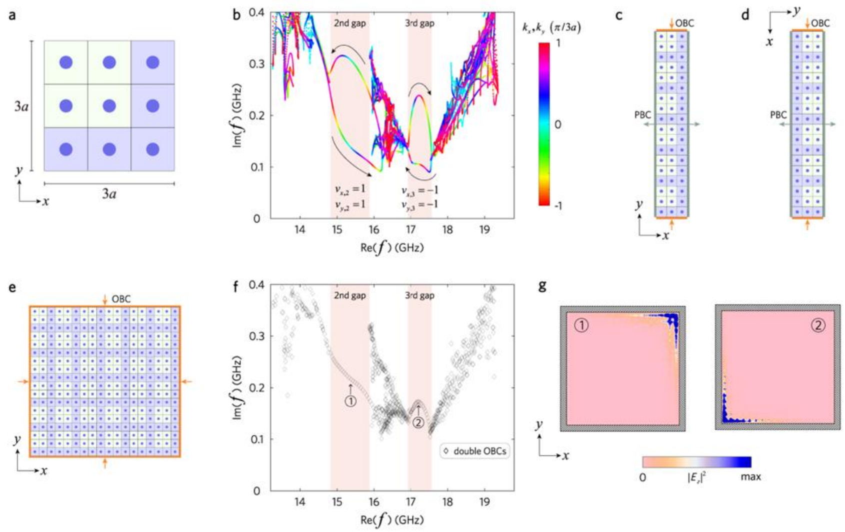
$d=2$ Line-gap topology		$d=1$ Point-gap topology	
H class	Invariant	NH class	Invariant (with $\mathcal{I}^\dagger$)
A ($\mathbb{Z}$)	$C \in \mathbb{Z}$	A ($\mathbb{Z}$)	$W(E_F) = C$ (mod 2)
D ($\mathbb{Z}$)	$C \in \mathbb{Z}$	D ($\mathbb{Z}_2$)	$\ldots$
		D† ($\mathbb{Z}$)	$W(0) = C$ (mod 2)
DIII ($\mathbb{Z}_2$)	$\nu \in \{0,1\}$	DIII† ($\mathbb{Z}_2$)	$\nu(0) = \nu$
AII ($\mathbb{Z}_2$)	$\nu \in \{0,1\}$	AII ($2\mathbb{Z}$)	$W(E_F) = 2\nu$ (mod 4)
		AII† ($\mathbb{Z}_2$)	$\nu(E_F) = \nu$
C ($2\mathbb{Z}$)	$C \in 2\mathbb{Z}$	C† ($2\mathbb{Z}$)	$W(0) = C$ (mod 4)

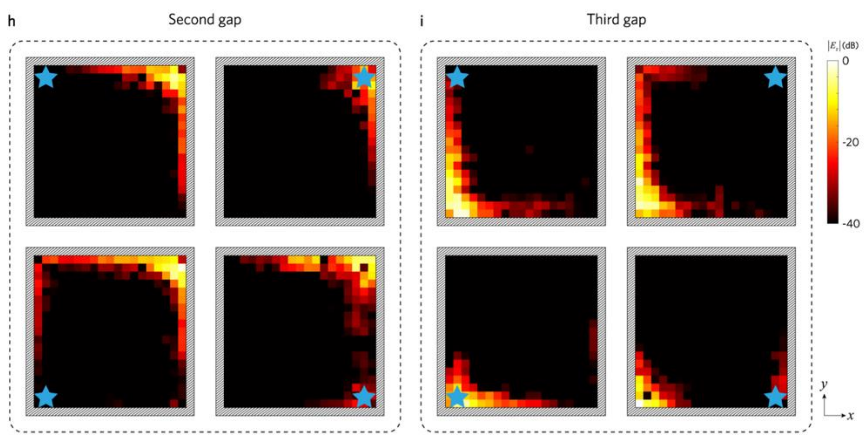
$d=3$ Line-gap topology		$d=2$ Point-gap topology	
H class	Invariant	NH class	Invariant (with $\mathcal{I}^\dagger$)
AIII ($\mathbb{Z}$)	$W \in \mathbb{Z}$	AIII ($\mathbb{Z}$)	$C(0) = W$ (mod 2)
DIII ($\mathbb{Z}$)	$W \in \mathbb{Z}$	DIII ($\mathbb{Z}_2$)	$\ldots$
		DIII† ($\mathbb{Z}$)	$C(0) = W$ (mod 2)
AII ($\mathbb{Z}_2$)	$\nu \in \{0,1\}$	AII† ($\mathbb{Z}_2$)	$\nu(E_F) = \nu$
CII ($\mathbb{Z}_2$)	$\nu \in \{0,1\}$	CII ($2\mathbb{Z}$)	$C(0) = 2\nu$ (mod 4)
		CII† ($\mathbb{Z}_2$)	$\nu(0) = \nu$
CI ($2\mathbb{Z}$)	$W \in 2\mathbb{Z}$	CI† ($2\mathbb{Z}$)	$C(0) = W$ (mod 4)

cf. Nakamura, Bessho & Sato, PRL **132**, 136401 (2024)

Experiments

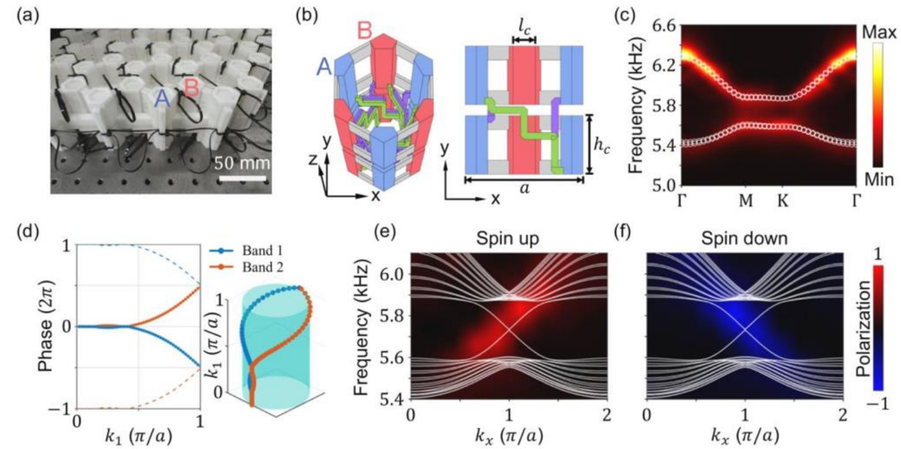
Photonic crystal

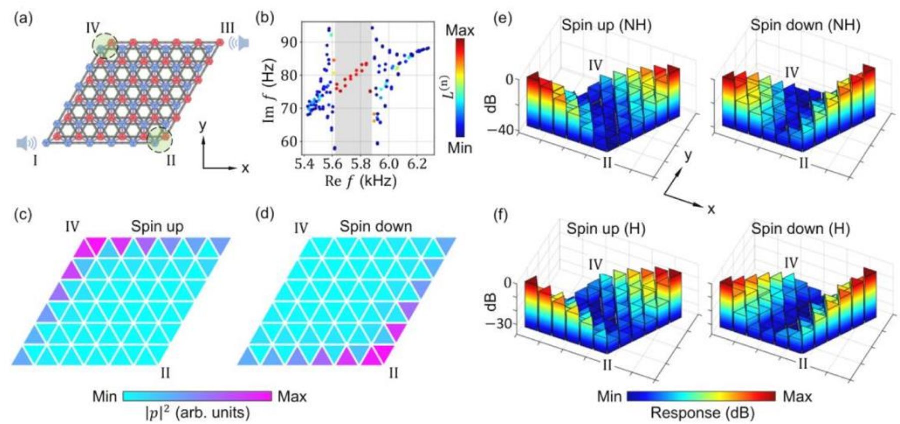




Liu *et al.*, Phys. Rev. Lett. **132**, 113802 (2024)

Phononic crystal



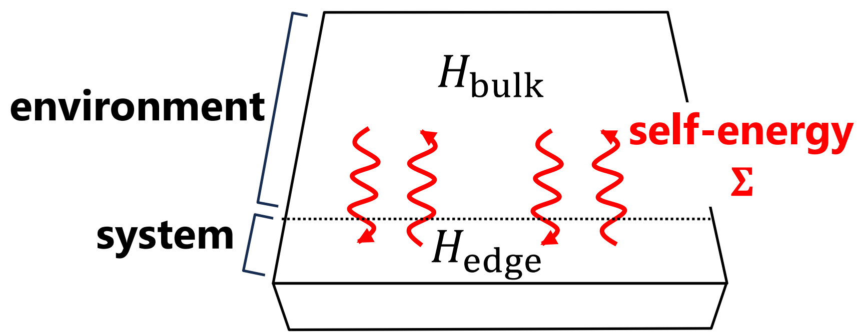


Wu *et al*. Phys. Rev. Lett. **133**, 126601 (2024)

So far, we have realized non-Hermitian topology by explicitly adding non-Hermiticity to the boundaries.

→ **We develop the Hermitian bulk – non-Hermitian boundary correspondence even in Hermitian topological insulators.**

(no dissipation)



Although no dissipation is added externally, the coupling between the bulk and boundary leads to non-Hermitian topology!

Outline

1. Introduction

2. Non-Hermitian topology (review)

3. Hermitian bulk – non-Hermitian boundary correspondence

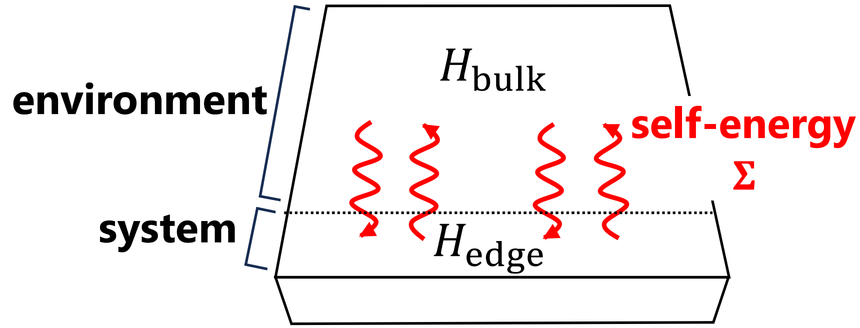
4. Non-Hermitian topology in Hermitian topological matter

5. Non-Hermitian origin of Wannier localizability and detachable topological boundary states

Non-Hermitian self-energy

Single-particle Hermitian Hamiltonian:

$$H = \begin{pmatrix} H_{\mathrm{bulk}} & T \\ T^{\dagger} & H_{\mathrm{edge}} \end{pmatrix}$$



Single-particle Schrödinger equation:

$$H \begin{pmatrix} |\psi_{\mathrm{bulk}}\rangle \\ |\psi_{\mathrm{edge}}\rangle \end{pmatrix} = (E + \mathrm{i}\eta) \begin{pmatrix} |\psi_{\mathrm{bulk}}\rangle \\ |\psi_{\mathrm{edge}}\rangle \end{pmatrix}$$

→ **Effective non-Hermitian Hamiltonian at the boundary:**

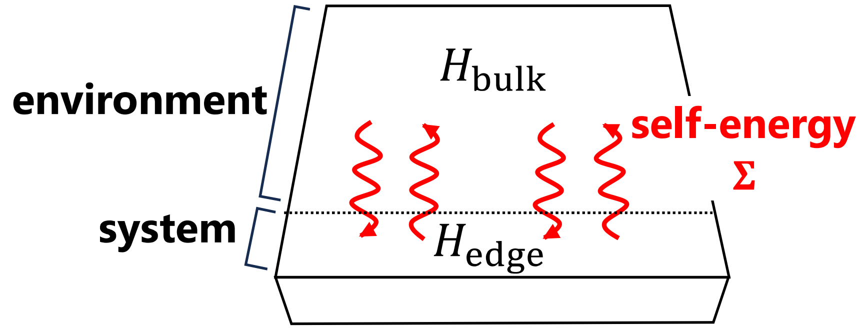
$$H_{\mathrm{eff}}(E) = H_{\mathrm{edge}} + \underline{\Sigma(E)}$$

self-energy $\Sigma(E) := T^{\dagger} (E + \mathrm{i}\eta - H_{\mathrm{bulk}})^{-1} T$

☆ Self-energy is generally non-Hermitian and captures the particle exchange between the bulk and boundary.

Non-Hermitian self-energy

Single-particle Hermitian Hamiltonian:



$$H = \begin{pmatrix} H_{\mathrm{bulk}} & T \\ T^{\dagger} & H_{\mathrm{edge}} \end{pmatrix}$$

Single-particle Schrödinger equation:

H

Topology of the original Hermitian Hamiltonian should leave an imprint on that of the effective non-Hermitian Hamiltonian at the boundary!

$$H_{\mathrm{eff}}\left(E\right) = H_{\mathrm{edge}} + \underline{\Sigma\left(E\right)}$$

self-energy $\Sigma\left(E\right) := T^{\dagger}\left(E + \mathrm{i}\eta - H_{\mathrm{bulk}}\right)^{-1} T$

☆ Self-energy is generally non-Hermitian and captures the particle exchange between the bulk and boundary.

Su-Schrieffer-Heeger model (1)

SSH model: $H_{\mathrm{SSH}}(k) = (v + t\cos k)\,\sigma_x + (t\sin k)\,\sigma_y$

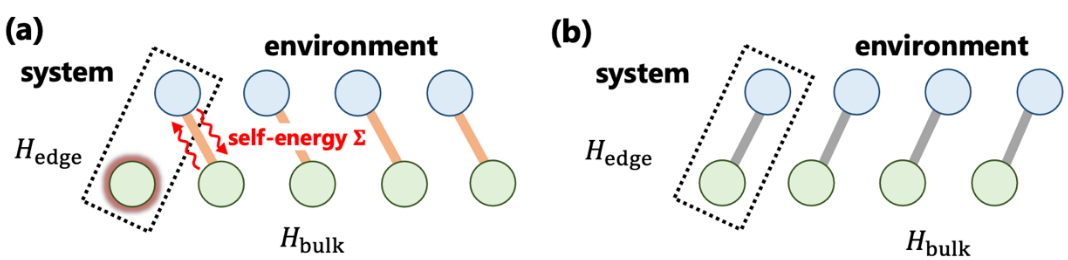
→ edge mode $|\psi_0\rangle \propto \sum_x \left(-\frac{v}{t}\right)^x |x\rangle \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ (top. phase: $|v/t| < 1$)

0D self-energy between the bulk and edge:

$$\Sigma(E) = \begin{pmatrix} 0 & 0 \\ 0 & -\mathrm{i}\pi\left(t^2 - v^2\right)\delta(E)\,\theta(t - v) \end{pmatrix}$$

chiral symmetry: $\sigma_z H_{\mathrm{SSH}}(k)\,\sigma_z = -H_{\mathrm{SSH}}(k)$

→ $\sigma_z H_{\mathrm{eff}}^{\dagger}(E=0)\,\sigma_z = -H_{\mathrm{eff}}(E=0)$

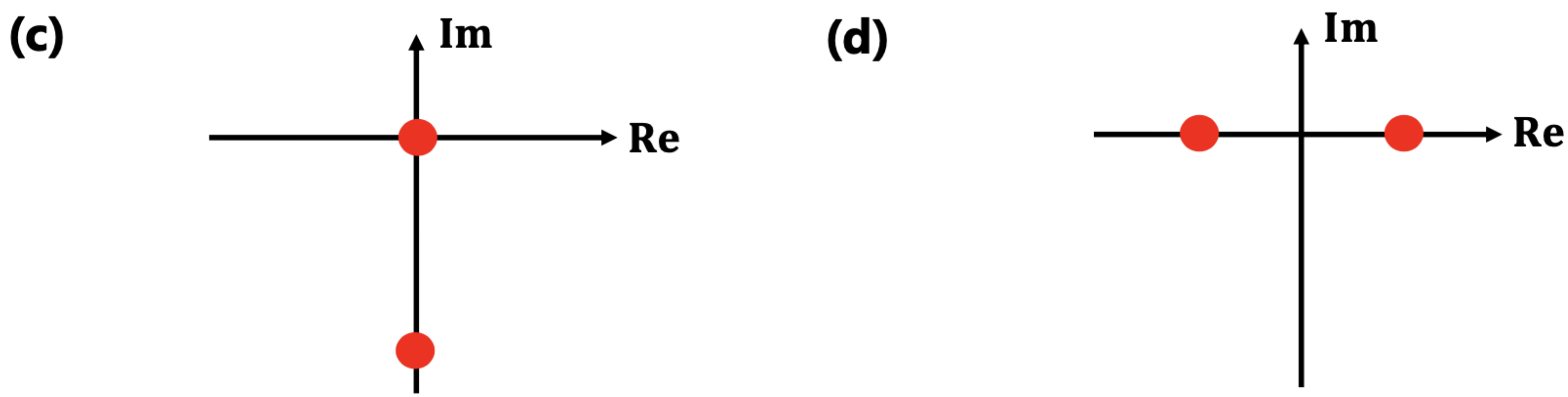


Su-Schrieffer-Heeger model (2)

$$\Sigma(E) = \begin{pmatrix} 0 & 0 \\ 0 & -i\pi\left(t^2 - v^2\right)\delta(E)\,\theta(t-v) \end{pmatrix}$$

topological

trivial



$$H_{\mathrm{eff}}(E=0) = v\sigma_x + \Sigma(E=0)$$

exhibits point-gap topology in 0D protected by chiral symmetry

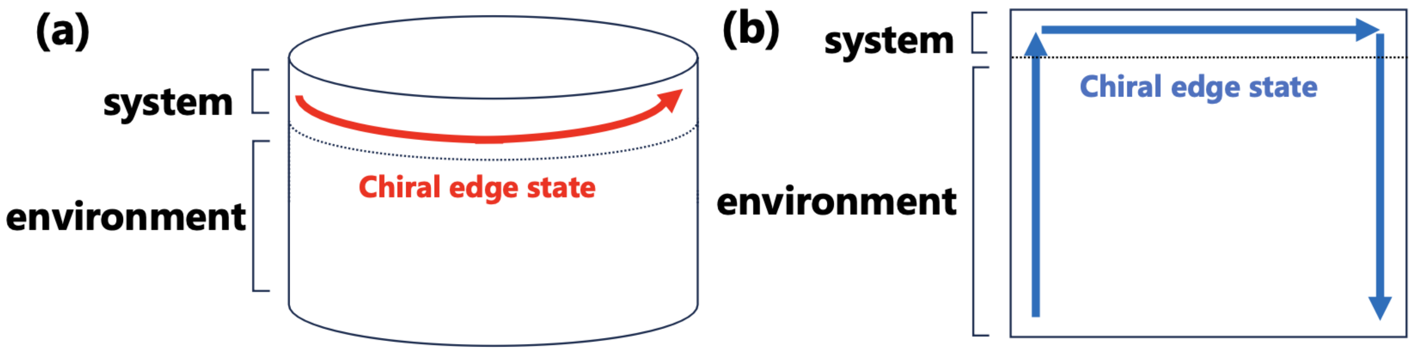
(0th Chern number of $iH_{\mathrm{eff}}\sigma_z$)

ensures the persistence of zero modes

Chern insulator (1)

$$H_{\mathrm{Chern}}\left(\boldsymbol{k}\right)=\left(t\sin k_x\right)\sigma_x+\left(t\sin k_y\right)\sigma_y+\left(m+t\cos k_x+t\cos k_y\right)\sigma_z$$

(1st) Chern number: $C_1=\begin{cases}\mathrm{sgn}\left(m/t\right) & \left(|m/t|<2\right)\\ 0 & \left(|m/t|>2\right)\end{cases}$



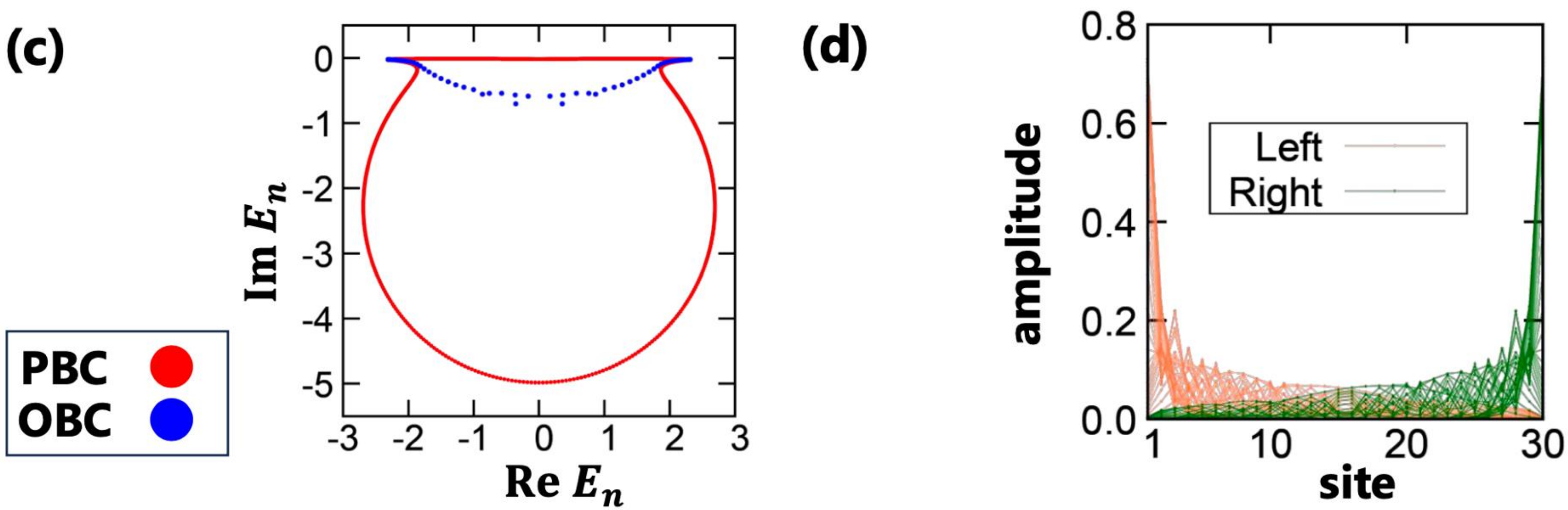
1D self-energy between the bulk and edge:

$$\Sigma\left(E,k_y\right)=\frac{t^2-\left(m+t\cos k_y\right)^2}{2\left(E+\mathrm{i}\eta-t\sin k_y\right)}\left(\sigma_0-\sigma_y\right)$$

$$H_{\mathrm{eff}}\left(E,k_y\right)=\left(t\sin k_y\right)\sigma_y+\left(m+t\cos k_y\right)\sigma_z+\Sigma\left(E,k_y\right)$$

Chern insulator (2)

The complex spectrum of H_{eff} exhibits winding.
(i.e., 1D point-gap topology)

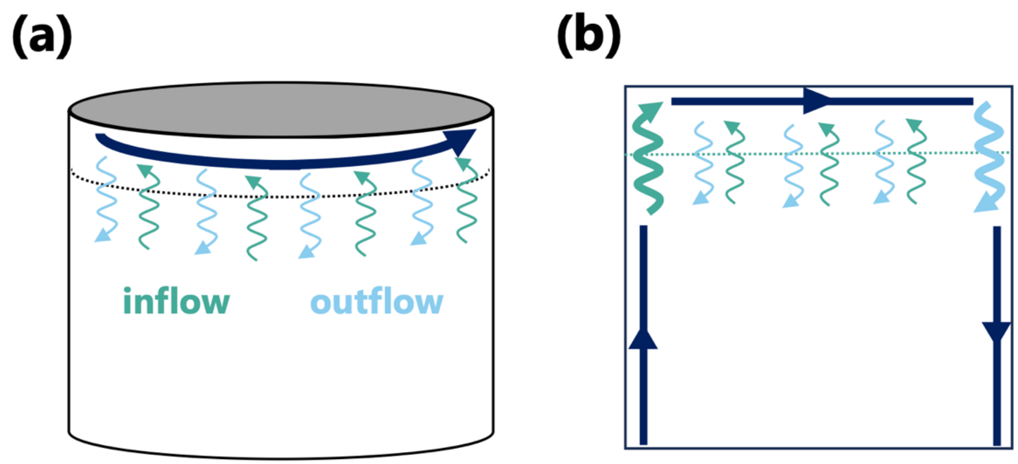


As a consequence of the point-gap topology,
H_{eff} also exhibits the non-Hermitian skin effect.
(localized at the corners)

Skin current (1)

What is a physical consequence of the point-gap topology and skin effect in the effective boundary non-Hermitian Hamiltonian?

→ **Skin effect of the chiral edge modes results in the localized current distributions.**

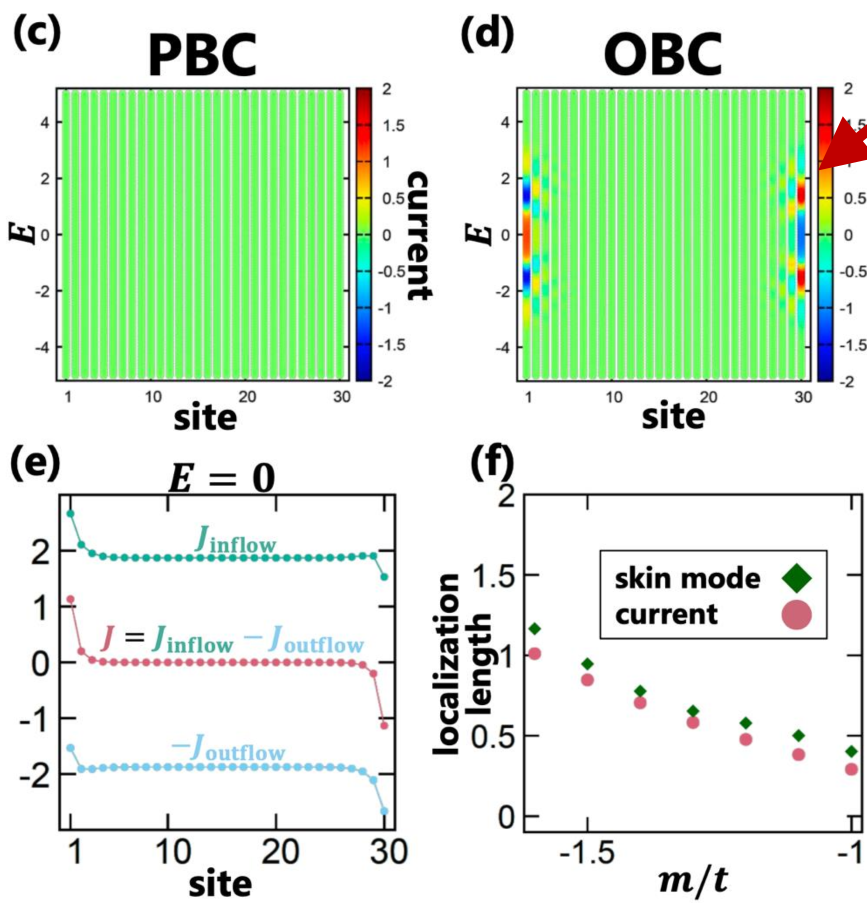


– **Inflow** from the bulk to the edge at one corner

– **Outflow** from the bulk to the edge at the other corner

Skin current (2)

E-resolved current: $J\left(E\right)=-\left[H_{\mathrm{edge}},G_{\mathrm{edge}}\left(E\right)\right]-\left[H_{\mathrm{edge}},G_{\mathrm{edge}}\left(E\right)\right]^{\dagger}$



Localized current arises only under the OBC

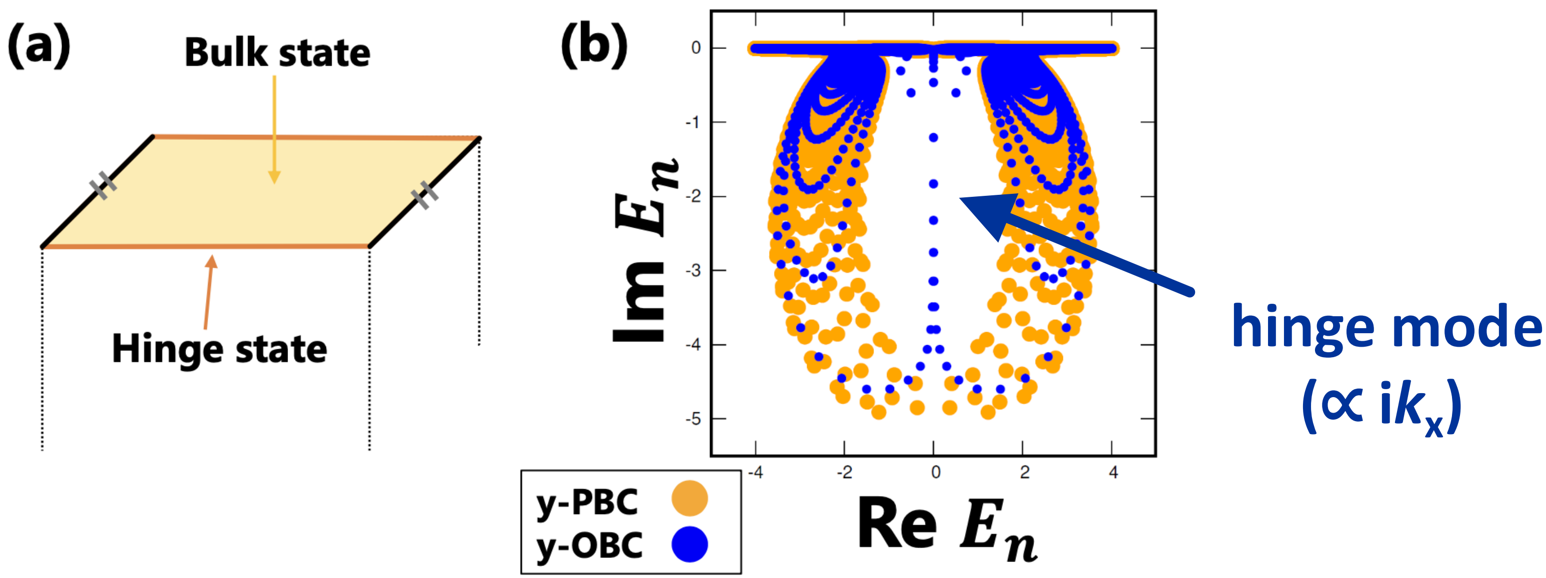
Localized current due to the skin effect

$$J_y \simeq j_1 e^{-y/\xi} - j_2 e^{-(L-y)/\xi}$$

ξ: localization length of skin mode

Three dimensions

3D topological insulator: $$H_{\mathrm{3DTI}}(\boldsymbol{k}) = (m + t\cos k_x + t\cos k_y + t\cos k_z)\,\tau_y + (t\sin k_x)\,\sigma_x\tau_x + (t\sin k_y)\,\sigma_y\tau_x + (t\sin k_z)\,\sigma_z\tau_x + \delta\,(\cos k_x + \cos k_y)\,\sigma_y\tau_y$$



H_{eff} exhibits 2D point-gap topology, leading to the chiral hinge modes!

(1st Chern number of $iH_{\mathrm{eff}}\sigma_z$)

Classification

When the original Hermitian Hamiltonians respect AZ symmetry, the effective non-Hermitian Hamiltonians H_{eff} respect AZ† symmetry.

→ **H_{eff} can generally exhibit point-gap topology**

skin effect

AZ class	TRS	PHS	CS	$d=1$	$d=2$	$d=3$
A	0	0	0	0	$\mathbb{Z}^*$	0
AIII	0	0	1	$\mathbb{Z}$	0	$\mathbb{Z}$
AI	$+1$	0	0	0	0	0
BDI	$+1$	$+1$	1	$\mathbb{Z}$	0	0
D	0	$+1$	0	$\mathbb{Z}_2$	$\mathbb{Z}^*$	0
DIII	-1	$+1$	1	$\mathbb{Z}_2$	$\mathbb{Z}_2^*$	$\mathbb{Z}^{**}$
AII	-1	0	0	0	$\mathbb{Z}_2^*$	$\mathbb{Z}_2^*$
CII	-1	-1	1	$2\mathbb{Z}$	0	$\mathbb{Z}_2$
C	0	-1	0	0	$2\mathbb{Z}^*$	0
CI	$+1$	-1	1	0	0	$2\mathbb{Z}$

hinge mode

cf. Nakamura, Bessho & Sato, PRL **132**, 136401 (2024)

cf. dynamical correspondence Lee *et al.*, PRL **123**, 206404 (2019); Bessho & Sato, PRL **127**, 196404 (2021)

Can we use our correspondence to understand Hermitian physics?

→ Yes, it quantifies the bulk-boundary coupling.

– **Intrinsic NH topology: Wannier obstructions**

– **Extrinsic NH topology: NO Wannier obstructions**

detachable boundary states

Using *K*-theory, we develop the classification of Wannier localizability and detachable boundary states in Hermitian topological insulators.

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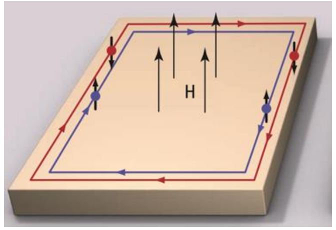
Wannier obstructions



Oh, Science **340**, 153 (2013)

Prototype of topological insulators: quantum Hall effect

Topological invariant: **Chern number**

Imposes obstructions to constructing exponentially localized Wannier functions

Vanderbilt, "Berry Phases in Electronic Structure Theory" (2018)

Fourier transform of Bloch wave functions

Localized electronic orbitals of crystalline materials

☆ The Wannier localizability provides a foundation of topological phases, including topological crystalline insulators

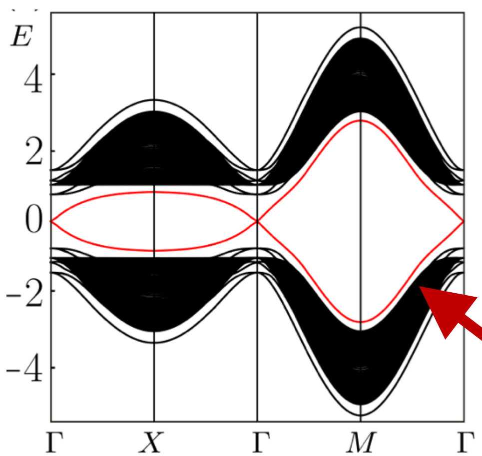
cf. Po *et al.*, PRL **121**, 126402 (2018)

Detachable topological boundary states

☆ **Not all topological insulators accompany Wannier obstructions!**

– 1D topological insulators   Kohn, PR **115**, 809 (1959)

– 3D topological insulators protected by chiral symmetry



Without Wannier obstructions, we can detach topological boundary states from the bulk bands!

detached!

Altland *et al.*, PRX **14**, 011057 (2024)

cf. Detachable topological boundary states in Hopf insulators

Alexandradinata *et al.*, PRB **103**, 045107 (2021)

Detachable topological boundary states

☆ **Not all topological insulators accompany Wannier obstructions!**

– 1D topological insulators Kohn, PR **115**, 809 (1959)

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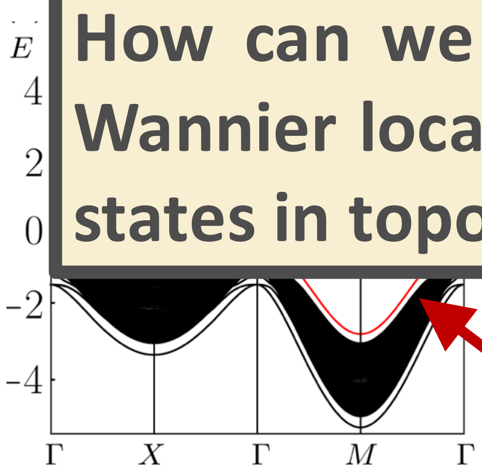


How can we generally understand and classify Wannier localizability and detachable boundary states in topological insulators?

can
from
the bulk bands!

detached!

Altland *et al.*, PRX **14**, 011057 (2024)

cf. Detachable topological boundary states in Hopf insulators

Alexandradinata *et al.*, PRB **103**, 045107 (2021)

Intrinsic vs extrinsic NH topology

☆ While some point-gap topology is intrinsic to non-Hermitian systems, others are continuously deformable to Hermitian/anti-Hermitian systems (extrinsic).

– 1D class A (intrinsic)

AZ class	Gap	Classifying space	$d=0$	$d=1$	$d=2$
A	P	$\mathcal{C}_1$	0	$\mathbb{Z}$	0
	L	$\mathcal{C}_0$	$\mathbb{Z}$	0	$\mathbb{Z}$

Unique to non-Hermitian systems

– 2D class AIII (extrinsic)

AZ class	Gap	Classifying space	$d=0$	$d=1$	$d=2$
AIII	P	$\mathcal{C}_0$	$\mathbb{Z}$	0	$\mathbb{Z}$
	$\mathrm{L_r}$	$\mathcal{C}_1$	0	$\mathbb{Z}$	0
	$\mathrm{L_i}$	$\mathcal{C}_0 \times \mathcal{C}_0$	$\mathbb{Z} \oplus \mathbb{Z}$	0	$\mathbb{Z} \oplus \mathbb{Z}$

Continuously deformable to anti-Hermitian systems

☆ We classify intrinsic/extrinsic point-gap topology.

Okuma, **Kawabata**, Shiozaki & Sato, PRL **124**, 086801 (2020); Shiozaki (in preparation)

Chiral edge states: intrinsic NH topology

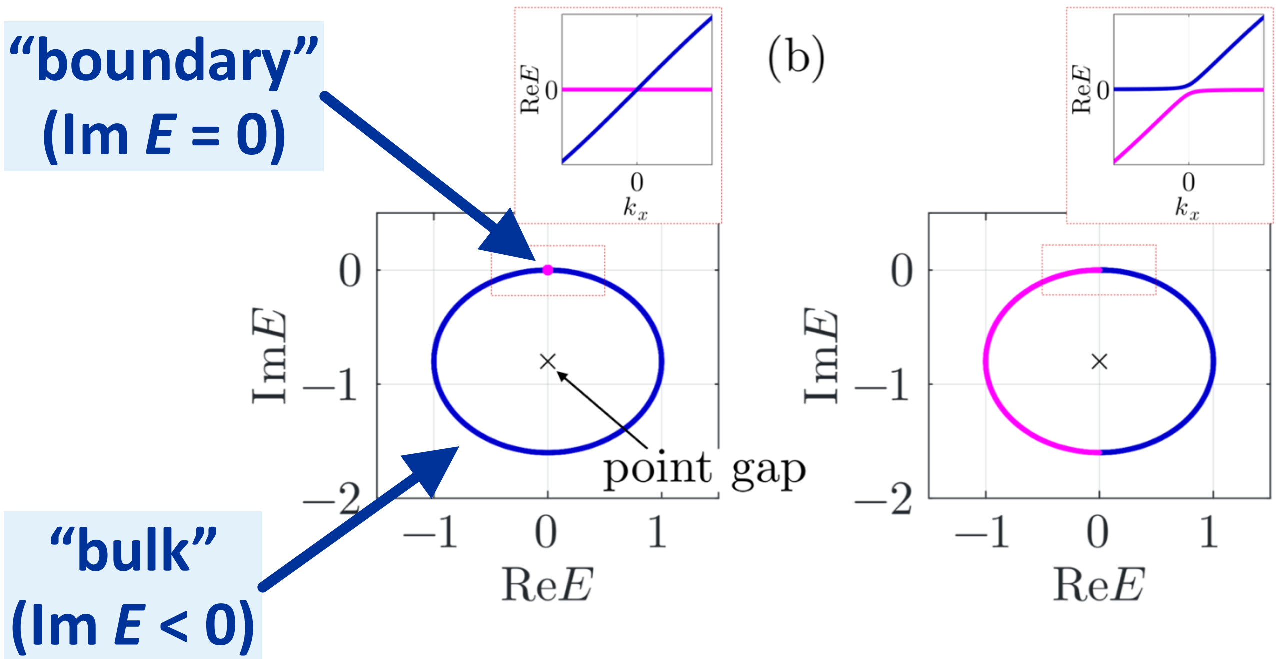
2D topological insulators (class A)

1D chiral edge state: $\mathcal{H}_{\mathrm{bdy}} = k_x$

→ **1D NH system:** $H = \sin k_x + \mathrm{i}\gamma \left(\cos k_x - 1 \right)$

Intrinsic: line-gap opening is prohibited

→ **Cannot be detached from the bulk bands!**



Dirac surface states: extrinsic NH topology (1)

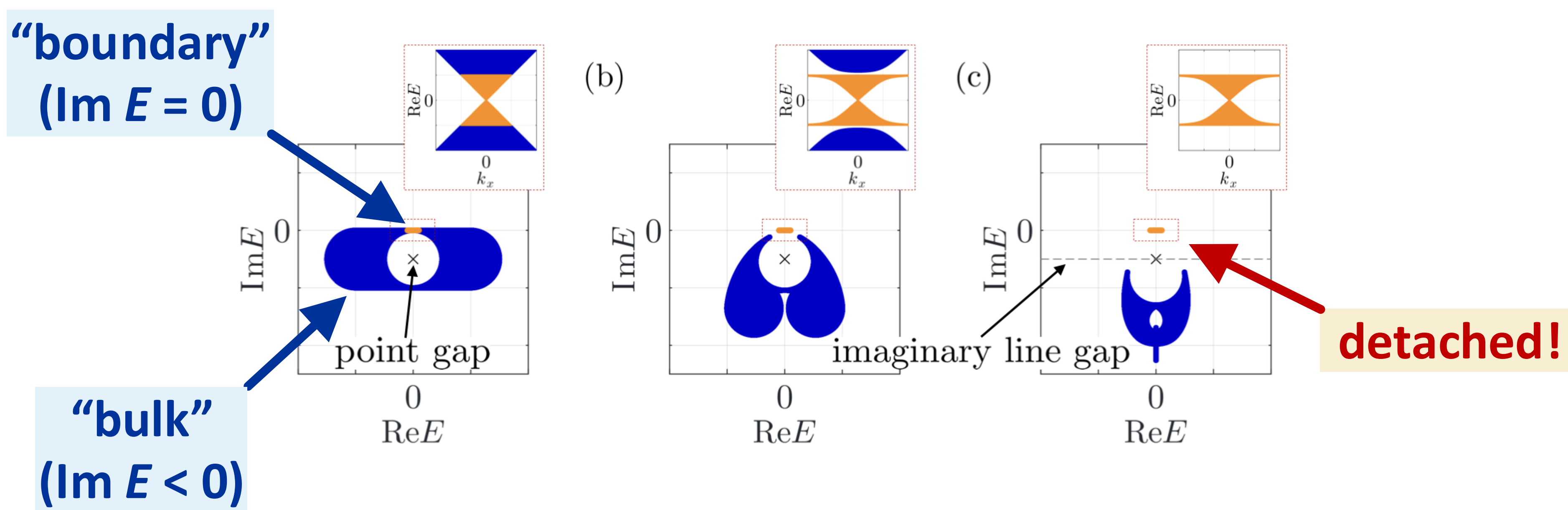
3D chiral-symmetric topological insulators (class AIII)

2D Dirac surface state: $\mathcal{H}_{\rm bdy} = k_x \sigma_x + k_y \sigma_y$

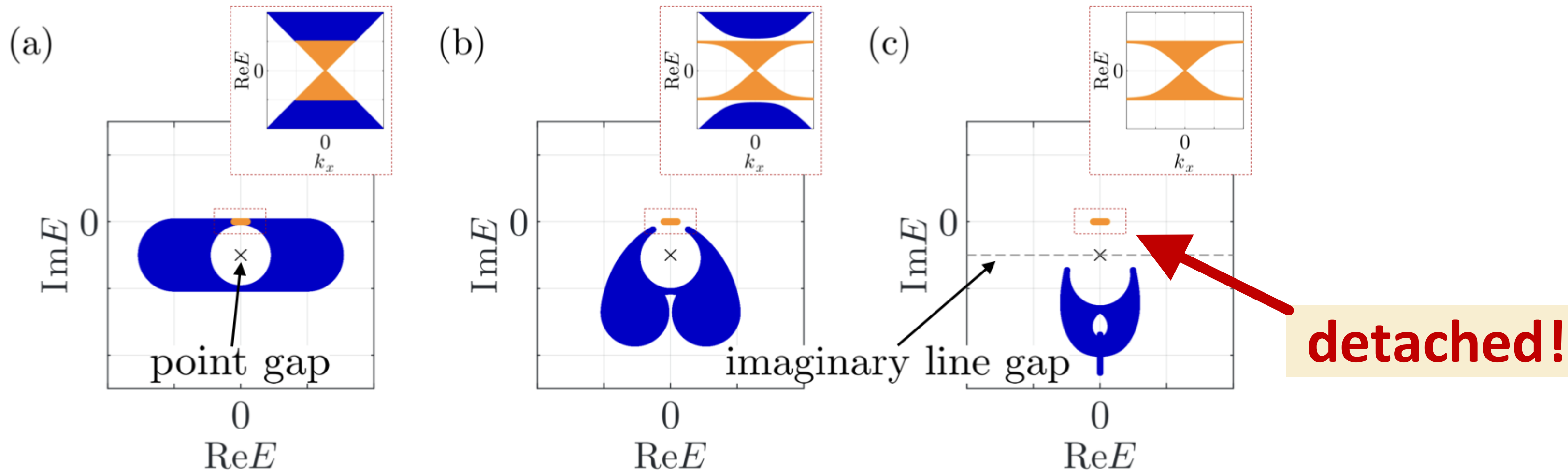
→ **2D NH system:** $H = (\sin k_x)\, \sigma_x + (1 - \cos k_x - \cos k_y)\, \sigma_y + \mathrm{i}\gamma\, (\sin k_y - 1)$

Extrinsic: continuously deformable to anti-Hermitian systems

→ **Detachable from the bulk bands!**



Dirac surface states: extrinsic NH topology (2)



AZ class	Gap	Classifying space	$d=0$	$d=1$	$\boxed{d=2}$
AIII	P	$\mathcal{C}_0$	$\mathbb{Z}$	0	$\mathbb{Z}$
	$\mathrm{L_r}$	$\mathcal{C}_1$	0	$\mathbb{Z}$	0
	$\mathrm{L_i}$	$\mathcal{C}_0 \times \mathcal{C}_0$	$\mathbb{Z} \oplus \mathbb{Z}$	0	$\mathbb{Z} \oplus \mathbb{Z}$

point-gap top.

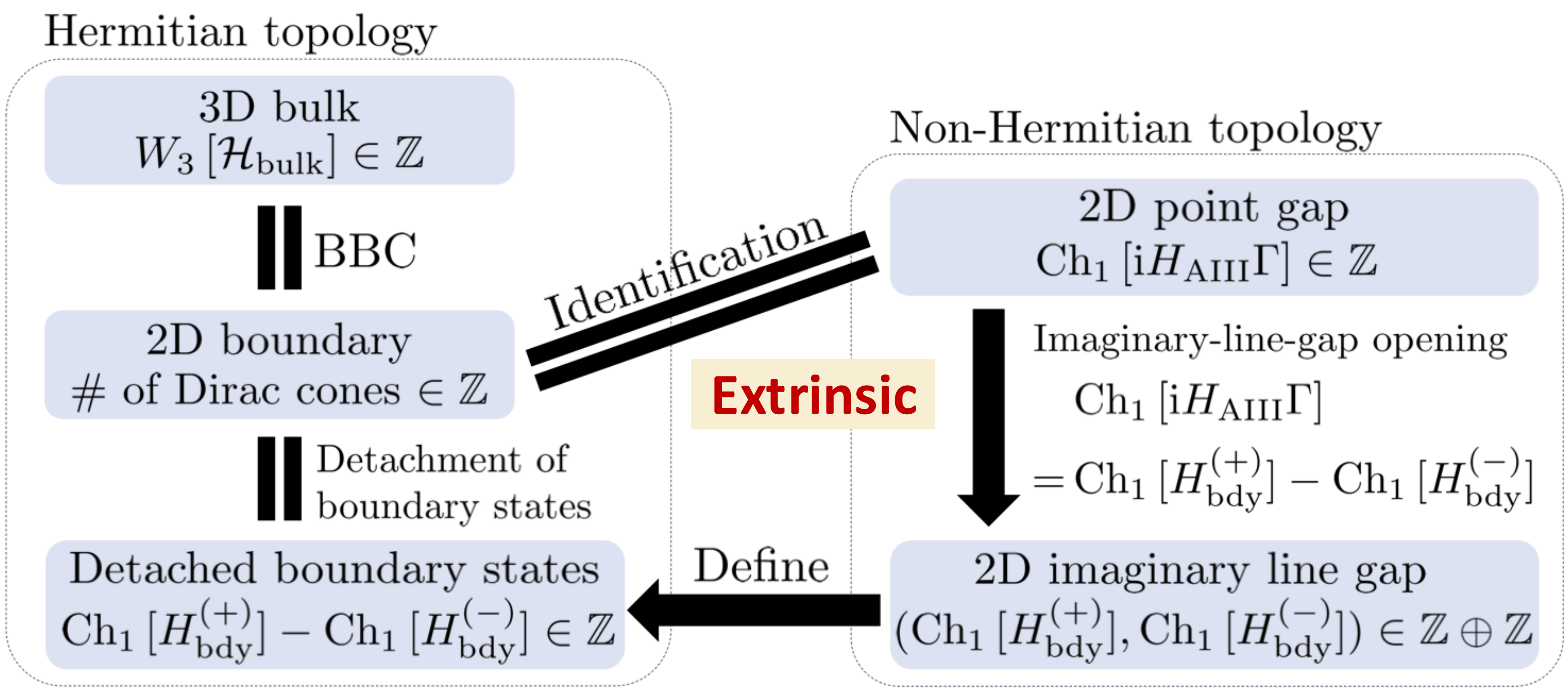
↓

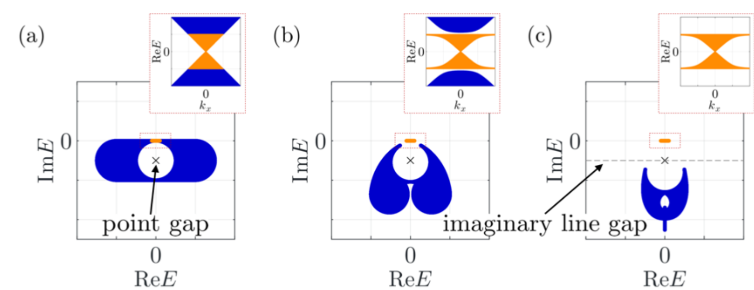
Imaginary-line-gap top.

(Chern number)

☆ Inheriting from point-gap topology, detached boundary states exhibit imaginary-line-gap topology!

Dirac surface states: extrinsic NH topology (3)





Classification (1)

☆ **We generally classify Wannier localizability and detachable topological boundary states based on non-Hermitian topology.**

(1) We identify Hermitian topological boundary states with non-Hermitian systems having **point-gap topology** (previous discussion).

(2) We classify whether point-gap topology is **intrinsic or extrinsic**.

– intrinsic: Wannier obstructions

– extrinsic: **NO** Wannier obstructions

detachable topological boundary states

→ Topology of detached boundary states is specified by **imaginary-line-gap topology**

Classification (2)

☆ Classification of Wannier localizability and detachable boundary states in Hermitian topological insulators

depends on symmetry classes

Class	TRS	PHS	CS	$d=1$	$d=2$	$d=3$	$d=4$	$d=5$	$d=6$	$d=7$	$d=8$
A	0	0	0	0	$\mathbb{Z}^{\checkmark}$	0	$\mathbb{Z}^{\checkmark}$	0	$\mathbb{Z}^{\checkmark}$	0	$\mathbb{Z}^{\checkmark}$
AIII	0	0	1	$\mathbb{Z}^{\times}$	0	$\mathbb{Z}^{\times}$	0	$\mathbb{Z}^{\times}$	0	$\mathbb{Z}^{\times}$	0
AI	$+1$	0	0	0	0	0	$2\mathbb{Z}^{\checkmark}$	0	$\mathbb{Z}_2^{\checkmark}$	$\mathbb{Z}_2^{\checkmark}$	$\mathbb{Z}^{\checkmark}$
BDI	$+1$	$+1$	1	$\mathbb{Z}^{\times}$	0	0	0	$2\mathbb{Z}^{\times}$	0	$\mathbb{Z}_2^{\times}$	$\mathbb{Z}_2^{\times}$
D	0	$+1$	0	$\mathbb{Z}_2^{\times}$	$\mathbb{Z}^{\checkmark}$	0	0	0	$2\mathbb{Z}^{\checkmark}$	0	$\mathbb{Z}_2^{\times}$
DIII	-1	$+1$	1	$\mathbb{Z}_2^{\times}$	$\mathbb{Z}_2^{\checkmark}$	$\mathbb{Z}^{\checkmark/\times}$	0	0	0	$2\mathbb{Z}^{\times}$	0
AII	-1	0	0	0	$\mathbb{Z}_2^{\checkmark}$	$\mathbb{Z}_2^{\checkmark}$	$\mathbb{Z}^{\checkmark}$	0	0	0	$2\mathbb{Z}^{\checkmark}$
CII	-1	-1	1	$2\mathbb{Z}^{\times}$	0	$\mathbb{Z}_2^{\times}$	$\mathbb{Z}_2^{\times}$	$\mathbb{Z}^{\times}$	0	0	0
C	0	-1	0	0	$2\mathbb{Z}^{\checkmark}$	0	$\mathbb{Z}_2^{\times}$	$\mathbb{Z}_2^{\times}$	$\mathbb{Z}^{\checkmark}$	0	0
CI	$+1$	-1	1	0	0	$2\mathbb{Z}^{\times}$	0	$\mathbb{Z}_2^{\times}$	$\mathbb{Z}_2^{\checkmark}$	$\mathbb{Z}^{\checkmark/\times}$	0

NO Wannier obstructions

cf. related classification: Lapierre, Trifunovic, Neupert & Brouwer, arXiv:2407.009757

Classification (3)

☆ Topological classification of detached boundary states (imaginary-line-gap topology)

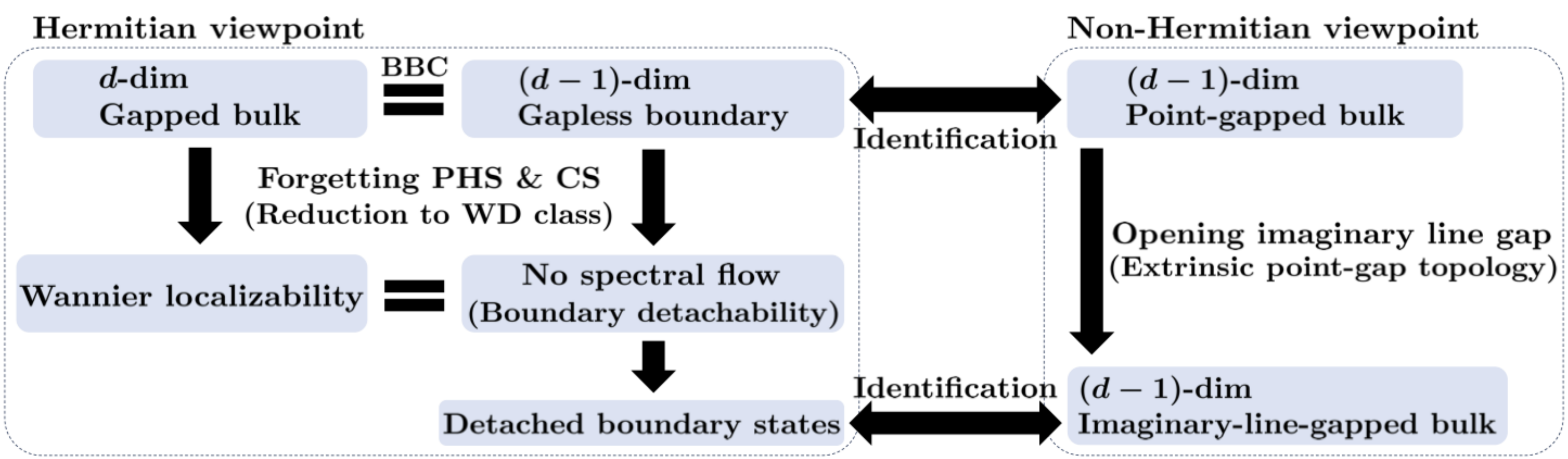
Class	$(d-1)$-dim. boundary top. inv. $=$ d-dim. bulk top. inv.
Chiral	$\mathrm{Ch}_n\,[H_{\mathrm{bdy}}^{(+)}] - \mathrm{Ch}_n\,[H_{\mathrm{bdy}}^{(-)}] = W_{2n+1}\,[\mathcal{H}_{\mathrm{bulk}}]$
BDI	$\nu_{\mathrm{AI}}^{d-1=8n+6}\,[H_{\mathrm{bdy}}] = \nu_{\mathrm{BDI}}^{d=8n+7}\,[\mathcal{H}_{\mathrm{bulk}}]$
BDI & D	$\nu_{\mathrm{AI}}^{d-1=8n+7}\,[H_{\mathrm{bdy}}] = \nu_{\mathrm{BDI\&D}}^{d=8n+8}\,[\mathcal{H}_{\mathrm{bulk}}]$
D	$\mathrm{Ch}_{4n}\,[H_{\mathrm{bdy}}] \equiv \nu_{\mathrm{D}}^{d=8n+1}\,[\mathcal{H}_{\mathrm{bulk}}] \pmod 2$
DIII	$\frac{1}{2}\mathrm{Ch}_{4n}\,[H_{\mathrm{bdy}}] \equiv \nu_{\mathrm{DIII}}^{d=8n+1}\,[\mathcal{H}_{\mathrm{bulk}}] \pmod 2$
CII	$\nu_{\mathrm{AII}}^{d-1=8n+2}\,[H_{\mathrm{bdy}}] = \nu_{\mathrm{CII}}^{d=8n+3}\,[\mathcal{H}_{\mathrm{bulk}}]$
C & CII	$\nu_{\mathrm{AII}}^{d-1=8n+3}\,[H_{\mathrm{bdy}}] = \nu_{\mathrm{C\&CII}}^{d=8n+4}\,[\mathcal{H}_{\mathrm{bulk}}]$
C	$\mathrm{Ch}_{4n+2}\,[H_{\mathrm{bdy}}] \equiv \nu_{\mathrm{C}}^{d=8n+5}\,[\mathcal{H}_{\mathrm{bulk}}] \pmod 2$
CI	$\frac{1}{2}\mathrm{Ch}_{4n+2}\,[H_{\mathrm{bdy}}] \equiv \nu_{\mathrm{CI}}^{d=8n+5}\,[\mathcal{H}_{\mathrm{bulk}}] \pmod 2$

K-theory approach

☆ We also formalize our classification with *K*-theory.

Shiozaki, Nakamura, Shimomura, Sato & **Kawabata**, arXiv:2407.18273

$$
\begin{tikzcd}
 & {}^{\phi}K_G^{(\tau,c_{\mathrm{triv}})+0}(T^{d-1}) \arrow[d, "f^{\mathrm{bdy}}_{\mathrm{L_i}}"] \\
{}^{\phi}K_G^{(\tau,c)+0}(T^d) \arrow[r, "f^{\mathrm{BBC}}_G"] \arrow[d, "f^{\mathrm{bulk}}_0"] & {}^{\phi}K_G^{(\tau,c)+1}(T^{d-1}) \arrow[d, "f^{\mathrm{bdy}}_0"] \\
{}^{\phi}K_{G_0}^{\tau+0}(T^d) \arrow[r, "f^{\mathrm{BBC}}_{G_0}"] & {}^{\phi}K_{G_0}^{\tau+1}(T^{d-1})
\end{tikzcd}
$$



Summary

PRX Quantum 4, 030315 (2023)

arXiv:2405.10015, 2407.09458, 2407.18273

・We establish the topological correspondence between the Hermitian bulk and non-Hermitian boundary.

・We develop this correspondence even in Hermitian topological insulators.

・We classify the Wannier localizability and detachable boundary states in Hermitian topological insulators.