Symmetry and Topology in Non-Hermitian Physics

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Pseudo-Hermitian Hamiltonians in Quantum Physics

December 9th, 2021

Collaborators

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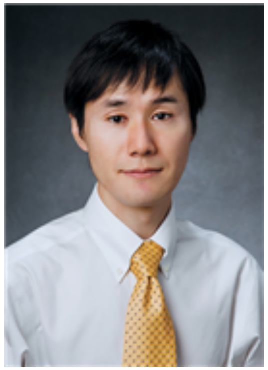
Masatoshi Sato
(Kyoto)



Shinsei Ryu
(Princeton)

Pseudo-Hermitian Hamiltonians in Quantum Physics

PT symmetry and pseudo-Hermicity are important for real spectra of non-Hermitian Hamiltonians.

→ **Other symmetry in non-Hermitian physics?**
(charge conjugation, chiral, …)

☆ **We develop general symmetry classification in non-Hermitian physics.**
(internal)

[Hermitian]		**[non-Hermitian]**
10-fold	→	**38-fold**
Altland & Zirnbauer, PRB **55**, 1142 (1997)		**KK**, Shiozaki, Ueda & Sato, PRX **9**, 041015 (2019)

Topological phases of matter

☆ **Topological phases:**

Phases of matter characterized by topology of the wavefunctions

- **The first example: integer quantum Hall effect (1980s)**

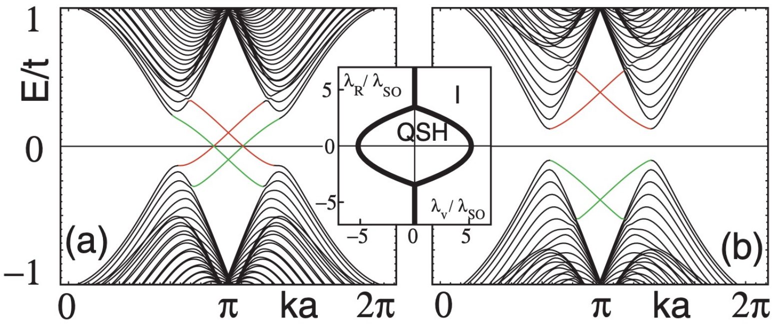
Phases are characterized by the **top. inv. (Chern #)** TKNN, PRL **49**, 405 (1982)

Robust gapless edge states appear (bulk-edge correspondence)

→ **Topological phases are everywhere, even in band theory!**

- **Z_2 topological insulator (TI)**

Quantum spin Hall effect (helical edge states)

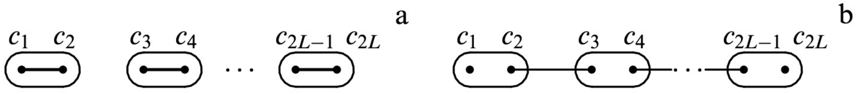


Kane & Mele, PRL **95**, 146802 (2005)

- **Topological superconductor (TSC)**

Majorana edge states

→ topological quantum computation?

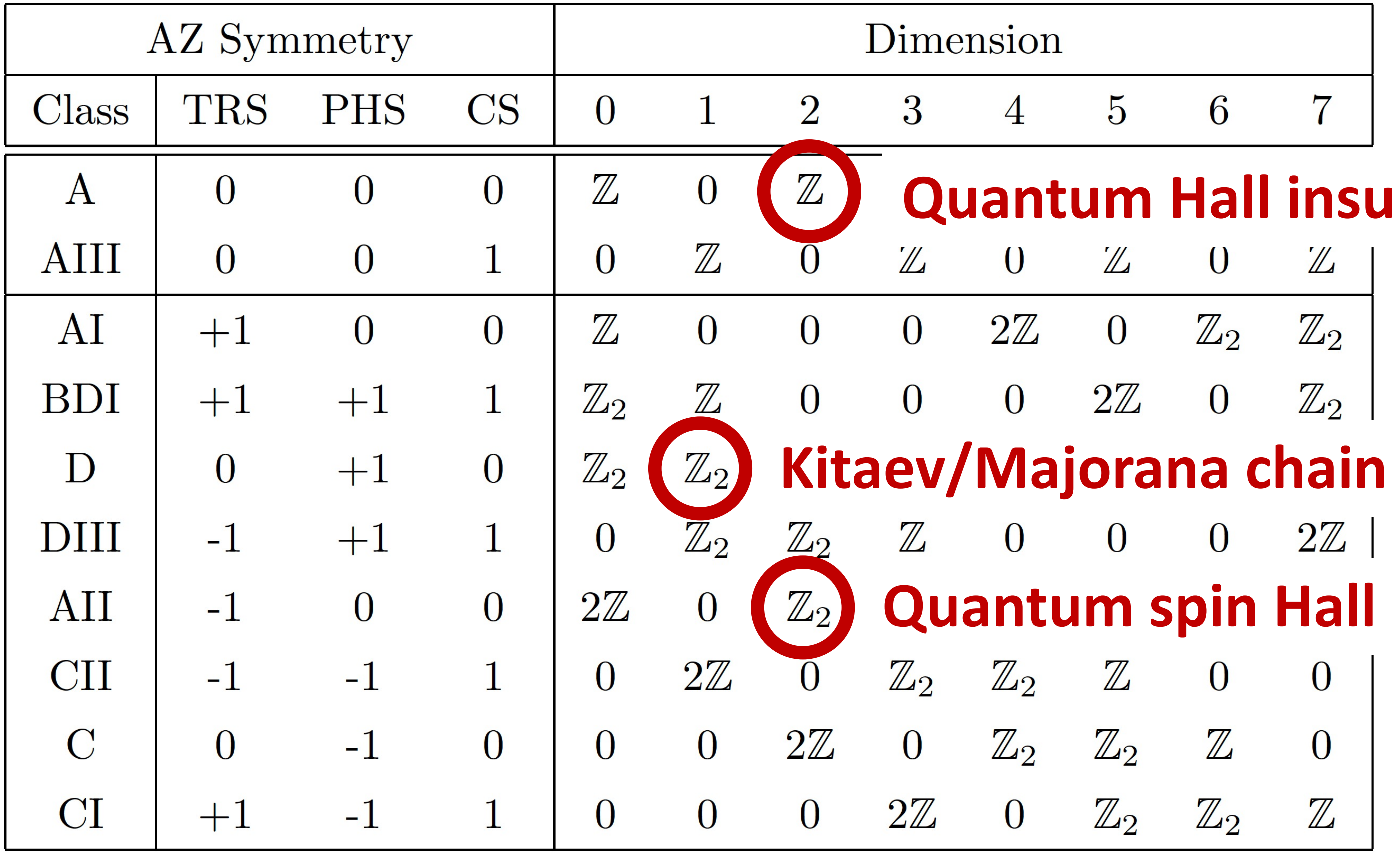


Kitaev, Phys.-Usp. **44**, 131 (2001)

Periodic table for TIs and TSCs

General and comprehensive theoretical framework of TIs and TSCs:

Periodic table based on spatial dimension and symmetry



AZ Symmetry				Dimension							
Class	TRS	PHS	CS	0	1	2	3	4	5	6	7
A	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$					
AIII	0	0	1	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AI	+1	0	0	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
BDI	+1	+1	1	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$
D	0	+1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$						
DIII	-1	+1	1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$
AII	-1	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$					
CII	-1	-1	1	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
C	0	-1	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
CI	+1	-1	1	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$

Quantum Hall insulator

Kitaev/Majorana chain

Quantum spin Hall insulator

Schnyder, Ryu, Furusaki & Ludwig, PRB **78**, 195125 (2008)

Kitaev, AIP Conf. Proc. **1134**, 22 (2009)

Non-Hermitian physics

Despite the enormous success, the existing framework of topological phases is **confined to Hermitian systems at equilibrium**.

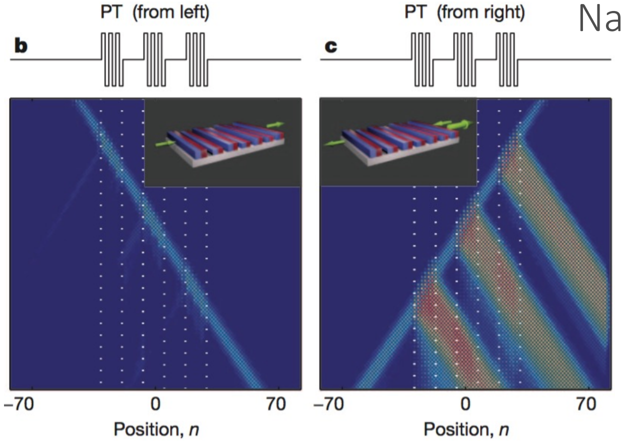
→ **Richer properties appear in non-Hermitian systems!**

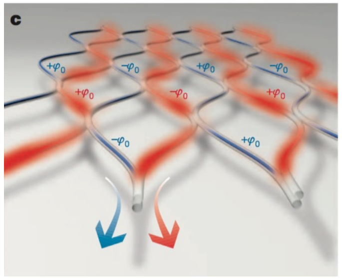
☆ Non-Hermiticity arises from **dissipation**, i.e., exchanges of energy or particles with an environment.

- **Photonic lattices with gain/loss**

Unidirectional light transport

Regensburger *et al.*, Nature **488**, 167 (2012)

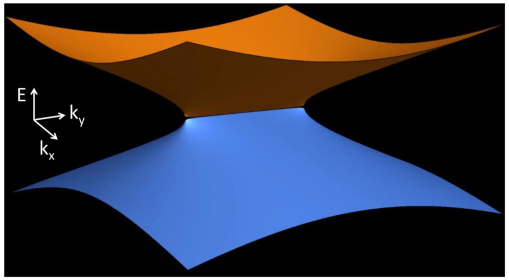


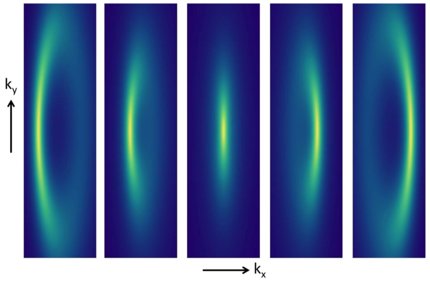


- **Finite-lifetime quasiparticles**

Bulk Fermi arc due to non-Hermitian self-energy

Kozii & Fu, arXiv: 1708.05841

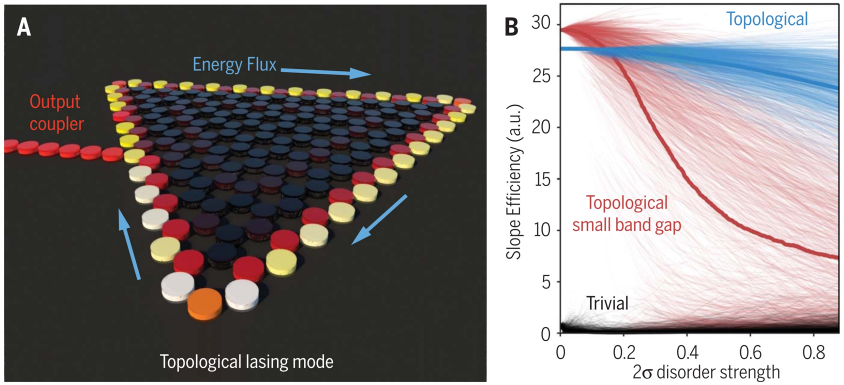




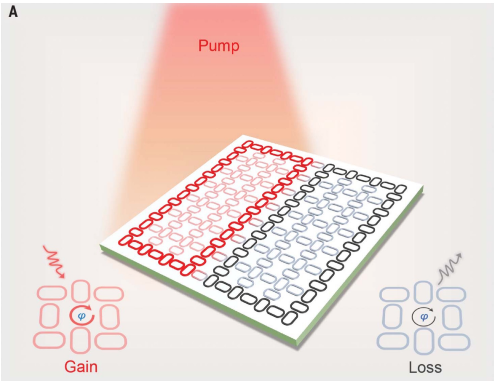
Non-Hermitian topological systems

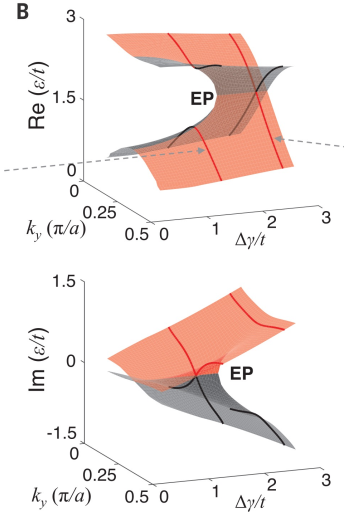
- **Topological laser**

New laser with high efficiency due to the **interplay of non-Hermiticity and topology**.



Bandres *et al.*, Science **359**, eaar4005 (2018)

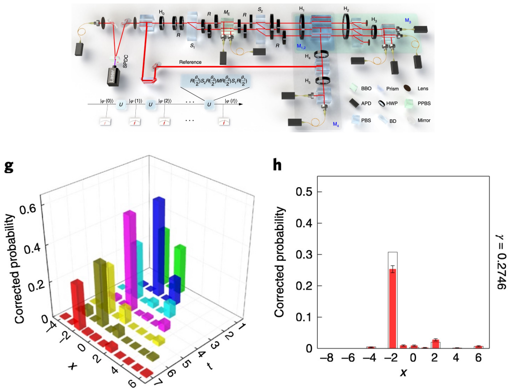




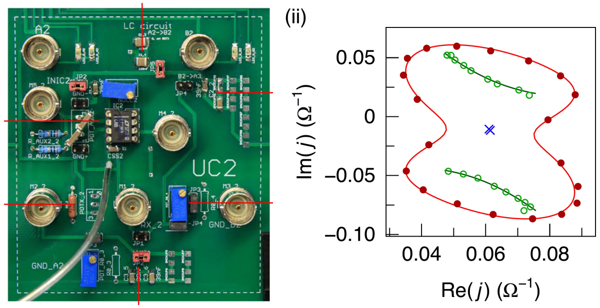
Zhao *et al.*, Science **365**, 1163 (2019)

- **Non-Hermitian skin effect**

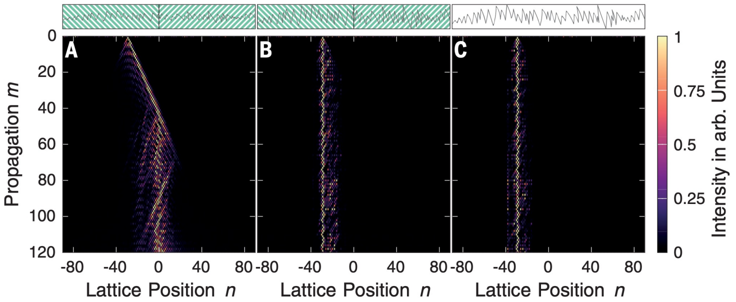
Anomalous localization of an extensive number of eigenstates due to non-Hermiticity.



Xiao *et al.*, Nat. Phys. **16**, 761 (2020)



Helbig *et al.*, Nat. Phys. **16**, 747 (2020)



Weidemann *et al.*, Science **368**, 311 (2020)

Non-Hermitian topological systems

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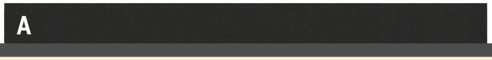
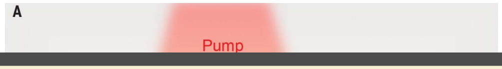
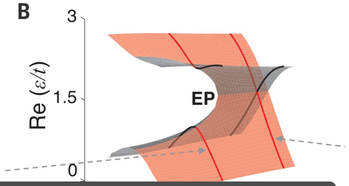
New laser with high efficiency due to the **interplay of non-Hermiticity and topology**.



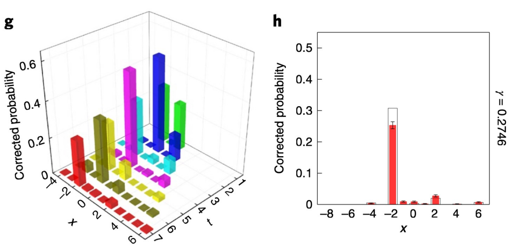




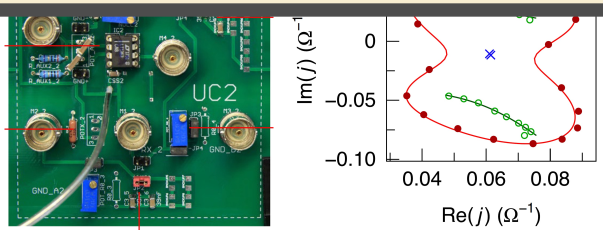
- While conventional physics focuses on Hermitian systems, **non-Hermiticity is natural and everywhere**.
- Unconventional phenomena occur as a result of the **interplay between non-Hermiticity and topology**.

non-Hermiticity ⟷ **topology**

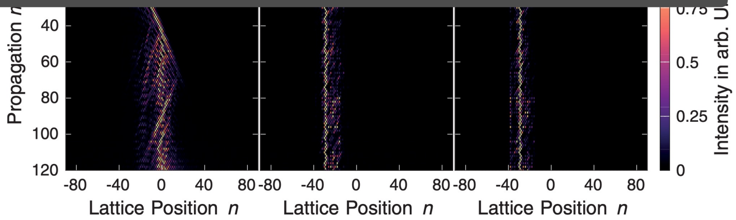
interplay



Xiao *et al.*, Nat. Phys. **16**, 761 (2020)



Helbig *et al.*, Nat. Phys. **16**, 747 (2020)



Weidemann *et al.*, Science **368**, 311 (2020)

☆ Non-Hermiticity gives rise to unique topological phases that have no Hermitian analogs.

Characterization? Bulk-boundary correspondence? Physical phenomena?

- **Classification**

  Two types of complex-energy gaps (point and line gaps)

- **Skin effect: new type of bulk-boundary correspondence**

  Intrinsic non-Hermitian topology leads to the skin effect

  Symmetry-protected skin effect

- **Topological field theory**

  Universal description of non-Hermitian topological phenomena

  The skin effect is a signature of a non-Hermitian anomaly

Outline

1. Introduction

2. Symmetry in non-Hermitian physics

3. Topology in non-Hermitian physics

4. Non-Hermitian skin effects

5. Topological field theory

Symmetry in Non-Hermitian Physics

Kawabata, Higashikawa, Gong, Ashida & Ueda, Nat. Commun. **10**, 297 (2019)

Kawabata, Shiozaki, Ueda & Sato, PRX **9**, 041015 (2019)

10-fold symmetry class

☆ 3-fold symmetry class by Wigner & Dyson

Wigner (1959)
Dyson, J. Math. Phys. **3**, 1199 (1962)

time reversal
$$\underset{\pm 1}{\underline{\mathcal{T}}} H^* \mathcal{T}^{-1} = H$$
anti-unitary
(with complex conjugation)

☆ 10-fold symmetry class by Altland & Zirnbauer

Altland & Zirnbauer,
PRB **55**, 1142 (1997)

particle hole
$$\mathcal{C} H^* \mathcal{C}^{-1} = -H$$
anti-unitary

chiral
(sublattice)
$$\Gamma H \Gamma^{-1} = -H$$
unitary

- **Universality**

Random matrix theory, topological classification, ……

Hermitian **Non-Hermitian**

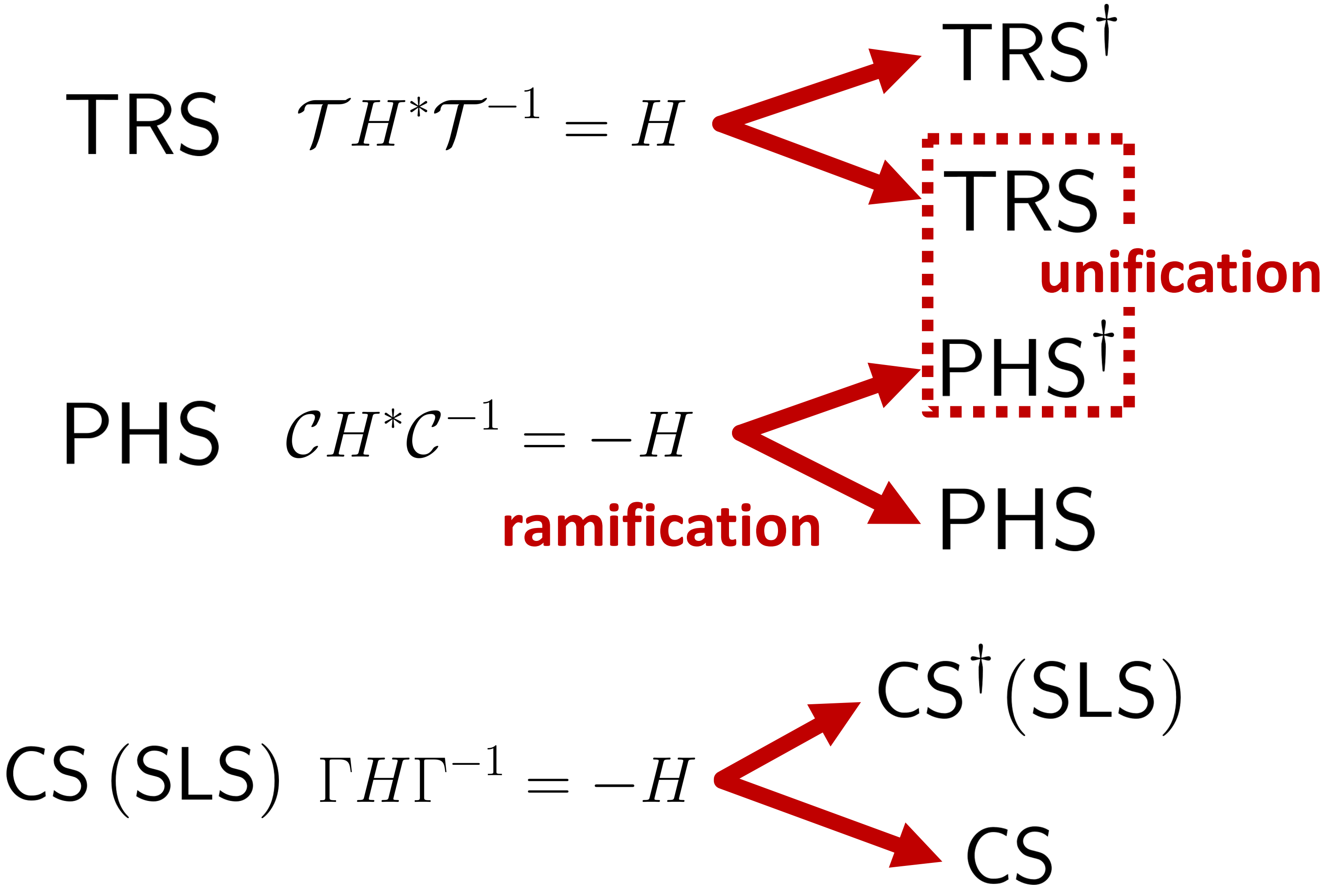
TRS $\mathcal{T} H^* \mathcal{T}^{-1} = H$ →

PHS $\mathcal{C} H^* \mathcal{C}^{-1} = -H$ → ?

CS (SLS) $\Gamma H \Gamma^{-1} = -H$ →

Hermitian

Non-Hermitian



TRS $\mathcal{T}H^*\mathcal{T}^{-1}=H$

PHS $\mathcal{C}H^*\mathcal{C}^{-1}=-H$

CS (SLS) $\Gamma H\Gamma^{-1}=-H$

$\mathrm{TRS}^\dagger$

TRS

unification

$\mathrm{PHS}^\dagger$

PHS

ramification

$\mathrm{CS}^\dagger$ (SLS)

CS

Symmetry ramification (1)

☆ Symmetry **ramifies (bifurcates)** in non-Hermitian systems.

KK, Shiozaki, Ueda & Sato,
PRX **9**, 041015 (2019).

Hermitian TRS **Non-Hermitian TRS**

time reversal

$$\begin{tikzcd}
\mathcal{T} H^* \mathcal{T}^{-1} = H \arrow[r] \arrow[d, equal, "H^* = H^T"] & \mathcal{T} H^* \mathcal{T}^{-1} = H \arrow[d, phantom, "\nparallel \;\; H^* \neq H^T"] \\
\mathcal{T} H^T \mathcal{T}^{-1} = H \arrow[r] & \mathcal{T} H^T \mathcal{T}^{-1} = H
\end{tikzcd}$$

reciprocity

Two types of symmetry appear due to the **distinction of complex conjugation and transpose operation** in non-Hermitian systems!

Symmetry ramification (2)

☆ Symmetry **ramifies (bifurcates)** in non-Hermitian systems.

KK, Shiozaki, Ueda & Sato,
PRX **9**, 041015 (2019).

Hermitian CS=SLS **Non-Hermitian CS/SLS**

sublattice

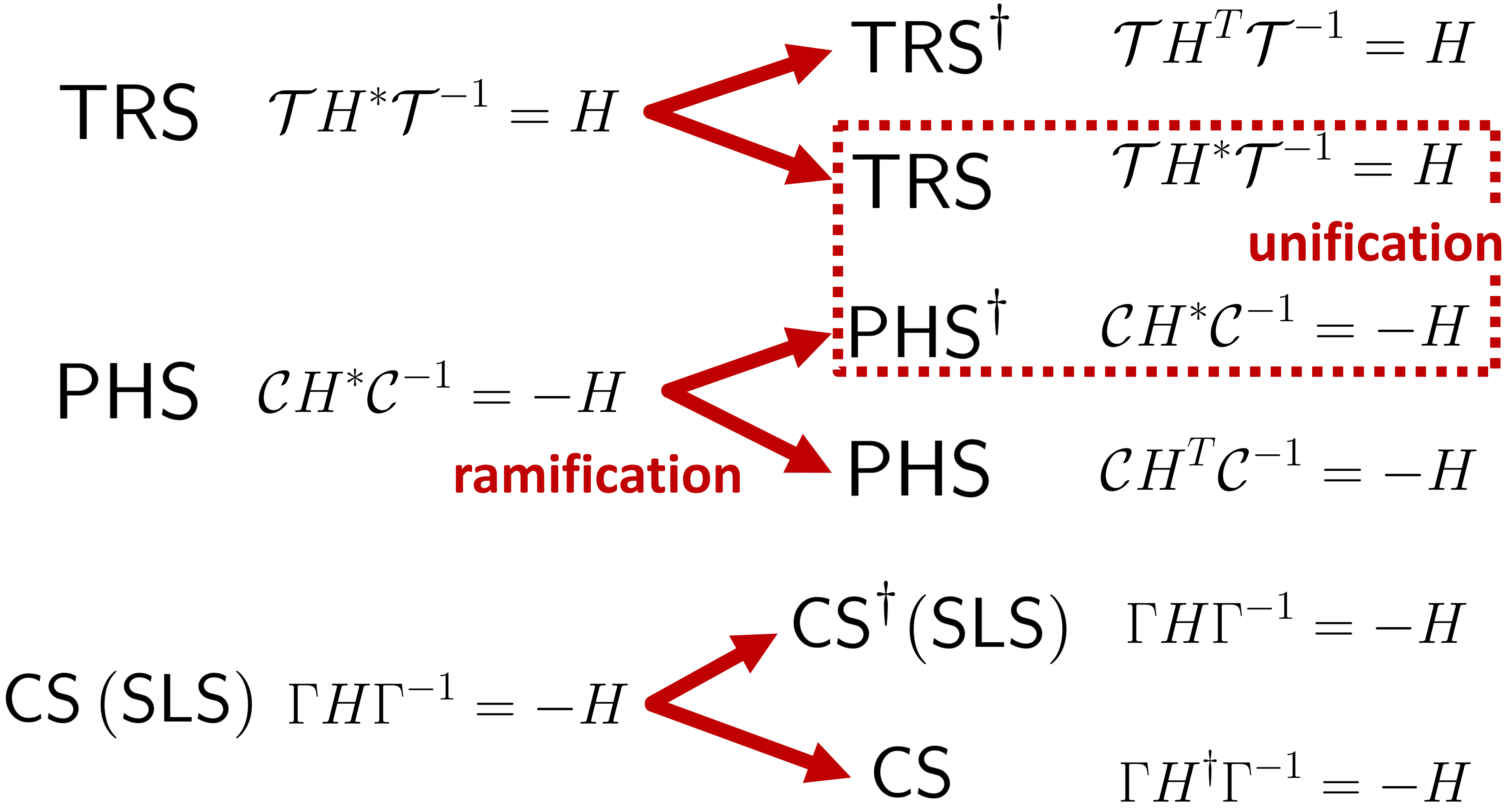
$$\begin{tikzcd}
\Gamma H \Gamma^{-1} = -H \arrow[r] \arrow[d, equal, "H = H^\dagger"] & \Gamma H \Gamma^{-1} = -H \arrow[d, equal, "/" marking, "H \neq H^\dagger"] \\
\Gamma H^\dagger \Gamma^{-1} = -H \arrow[r] & \Gamma H^\dagger \Gamma^{-1} = -H
\end{tikzcd}$$

chiral

Two types of symmetry appear due to the **distinction of complex conjugation and transpose operation** in non-Hermitian systems!

Hermitian

Non-Hermitian



TRS $\mathcal{T}H^*\mathcal{T}^{-1} = H$

TRS† $\mathcal{T}H^T\mathcal{T}^{-1} = H$

TRS $\mathcal{T}H^*\mathcal{T}^{-1} = H$

unification

PHS† $\mathcal{C}H^*\mathcal{C}^{-1} = -H$

PHS $\mathcal{C}H^*\mathcal{C}^{-1} = -H$

ramification

PHS $\mathcal{C}H^T\mathcal{C}^{-1} = -H$

CS (SLS) $\Gamma H\Gamma^{-1} = -H$

CS† (SLS) $\Gamma H\Gamma^{-1} = -H$

CS $\Gamma H^\dagger\Gamma^{-1} = -H$

Symmetry unification (1)

☆ Antiunitary symmetries distinct in Hermitian systems are **unified** in non-Hermitian systems.

KK *et al.*, Nat. Commun. **10**, 297 (2019)

$$\mathcal{T} H^{*} \mathcal{T}^{-1} = H$$

$$\mathcal{C} H^{*} \mathcal{C}^{-1} = -H$$

Two antiunitary symmetries are distinct for **Hermitian** H

→ If we allow **non-Hermitian** H, they are equivalent!

$$\mathcal{T} H^{*} \mathcal{T}^{-1} = H \longleftrightarrow \mathcal{T} \left[\mathrm{i} H \right]^{*} \mathcal{T}^{-1} = \underline{\underline{-}} \left[\mathrm{i} H \right]$$

one-to-one mapping

(wavefunctions are invariant)

Symmetry unification (2)

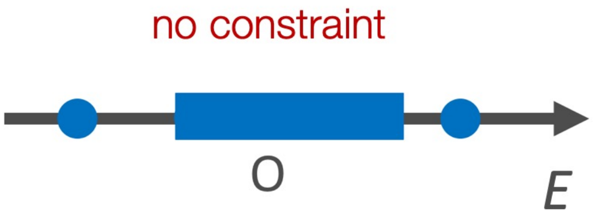
☆ Antiunitary symmetries distinct in Hermitian systems are **unified** in non-Hermitian systems.

KK *et al.*, Nat. Commun. **10**, 297 (2019)

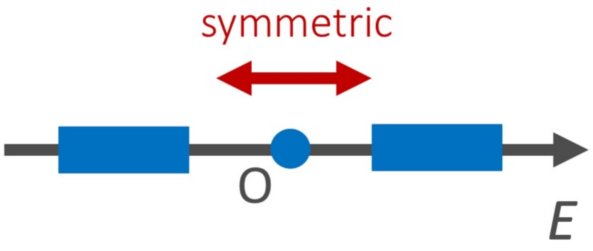
Symmetry	Hermitian	Non-Hermitian
$\mathcal{T}$	no constraints	$E \in \mathbb{R}$ or $(E,\ E^*)$
$\mathcal{C}$	$E = 0$ or $(E,\ -E)$	$E \in \mathrm{i}\mathbb{R}$ or $(E,\ -E^*)$

(Hermitian)

$\mathcal{T}$



$\mathcal{C}$



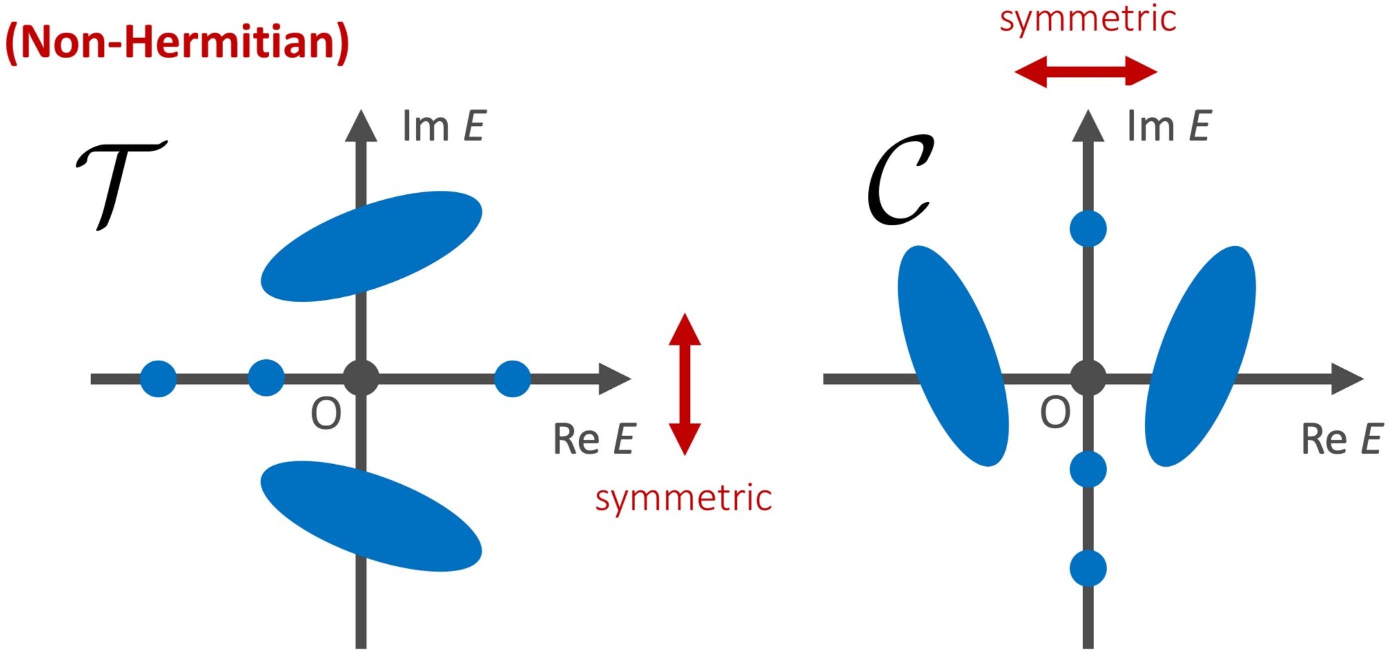
Symmetry unification (2)

☆ Antiunitary symmetries distinct in Hermitian systems are **unified** in non-Hermitian systems.

KK *et al.*, Nat. Commun. **10**, 297 (2019)

Symmetry	Hermitian	Non-Hermitian
$\mathcal{T}$	no constraints	$E \in \mathbb{R}$ or $(E,\ E^*)$
$\mathcal{C}$	$E = 0$ or $(E,\ -E)$	$E \in \mathrm{i}\mathbb{R}$ or $(E,\ -E^*)$

(Non-Hermitian)



38-fold symmetry class

- Hermitian case: **10 classes** (AZ symmetry class)

  time reversal, particle hole, and chiral (=sublattice)

- Non-Hermitian case: **38 classes**

KK, Shiozaki, Ueda & Sato,
PRX **9**, 041015 (2019).

10-fold non-Hermitian AZ symmetry class

$$\mathcal{T}_+ H^* \mathcal{T}_+ = H, \;\; \mathcal{C}_- H^T \mathcal{C}_- = -H, \;\; \Gamma H^\dagger \Gamma = -H$$

10-fold non-Hermitian $\mathrm{AZ}^\dagger$ symmetry class

$$\mathcal{C}_+ H^T \mathcal{C}_+ = H, \;\; \mathcal{T}_- H^* \mathcal{T}_- = -H, \;\; \Gamma H^\dagger \Gamma = -H$$

(**Hermitian conjugate** of the AZ class)

22-fold non-Hermitian AZ symmetry class with **sublattice symmetry**

$$\mathcal{S} H \mathcal{S} = -H$$ (**NOT** equivalent to chiral symmetry)

10 + 10 + 22 – 4 = 38 symmetry classes

$$\underbrace{-4}_{\text{unification}}$$

38-fold symmetry class

New symmetry classes lead to new physics:

- **Non-Hermitian random matrix theory**

  Hamazaki, **KK**, Kura & Ueda, PRR **2**, 023286 (2020)

- **Non-Hermitian Anderson localization**

  KK & Ryu, PRL **126**, 166801 (2021)

- **Non-Hermitian topological phases**

  (gapped) **KK**, Shiozaki, Ueda & Sato, PRX **9**, 041015 (2019)

  (gapless) **KK**, Bessho & Sato, PRL **123**, 066405 (2019)

- **Non-Hermitian skin effects**

  Okuma, **KK**, Shiozaki & Sato, PRL **124**, 086801 (2020)

$10 + 10 + 22 \underbrace{- 4}_{\text{unification}} = 38$ symmetry classes

Topology in Non-Hermitian Physics

Gong, Ashida, **Kawabata**, Takasan, Higashikawa & Ueda, PRX **8**, 031079 (2019)

Kawabata, Shiozaki, Ueda & Sato, PRX **9**, 041015 (2019)

Kawabata, Bessho & Sato, PRL **123**, 066405 (2019)

Complex-energy gaps (1)

An “**energy gap**” is needed to define a topological phase.

However, a non-Hermitian extension of an **“energy gap”** is nontrivial since the **spectrum is complex**.

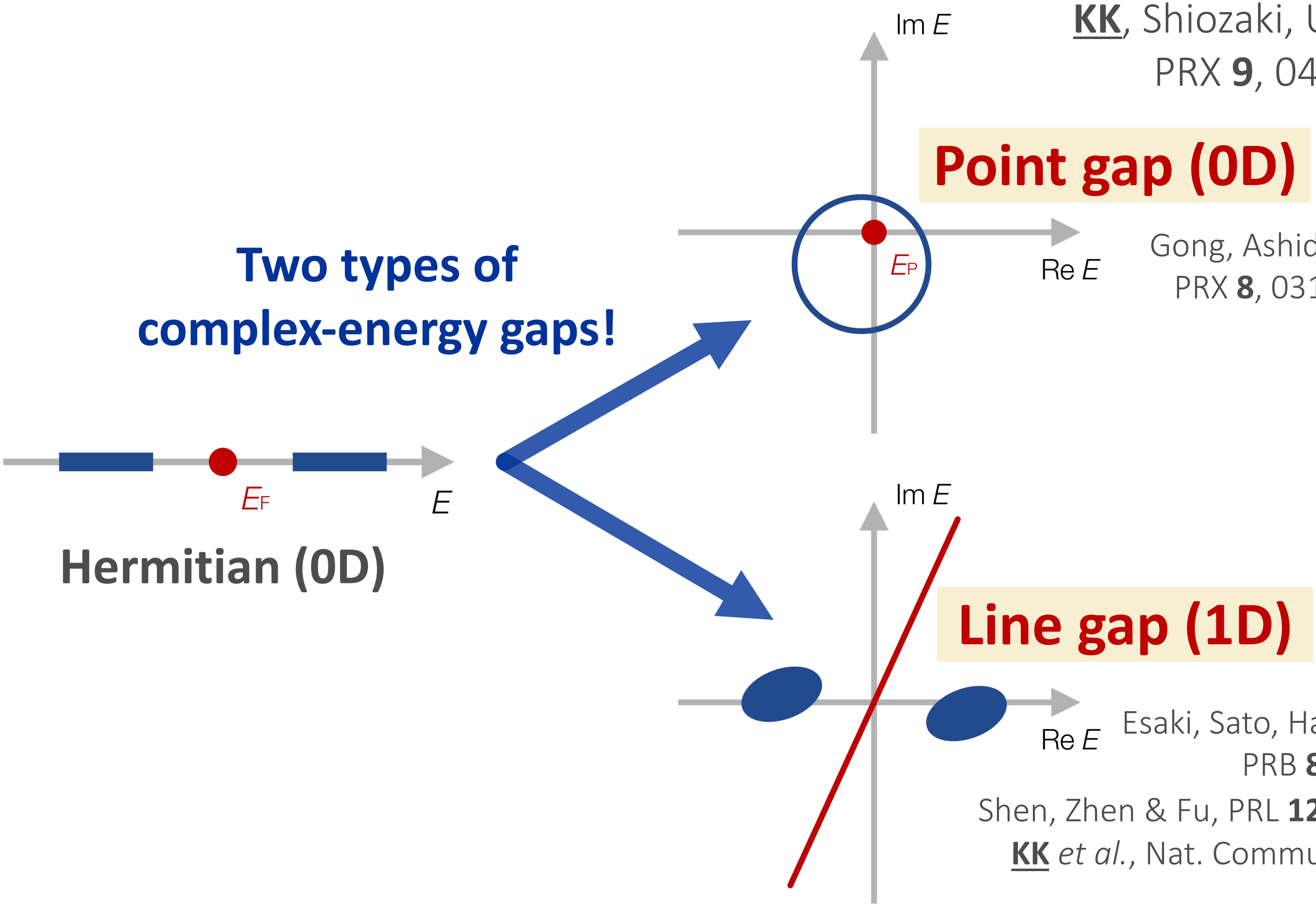
Hermitian

Energy gap in Hermitian systems:

- Energy regions where states are **forbidden to be present**.
- They **should be point-like (0D)** since the **real spectrum is 1D**.

→ Since the **complex spectrum is 2D (real and imaginary)**, such vacant regions can be **either 0D or 1D**!

Complex-energy gaps (2)



Two types of complex-energy gaps!

Hermitian (0D)

Point gap (0D)

KK, Shiozaki, Ueda & Sato, PRX **9**, 041015 (2019)

Gong, Ashida, **KK** *et al.*, PRX **8**, 031079 (2018)

Line gap (1D)

Esaki, Sato, Hasebe & Kohmoto, PRB **84**, 205128 (2011)
Shen, Zhen & Fu, PRL **120**, 146402 (2018)
KK *et al.*, Nat. Commun. **10**, 297 (2019)

NOTE: The definition that should be adopted depends on the individual physical situations that we are interested in.

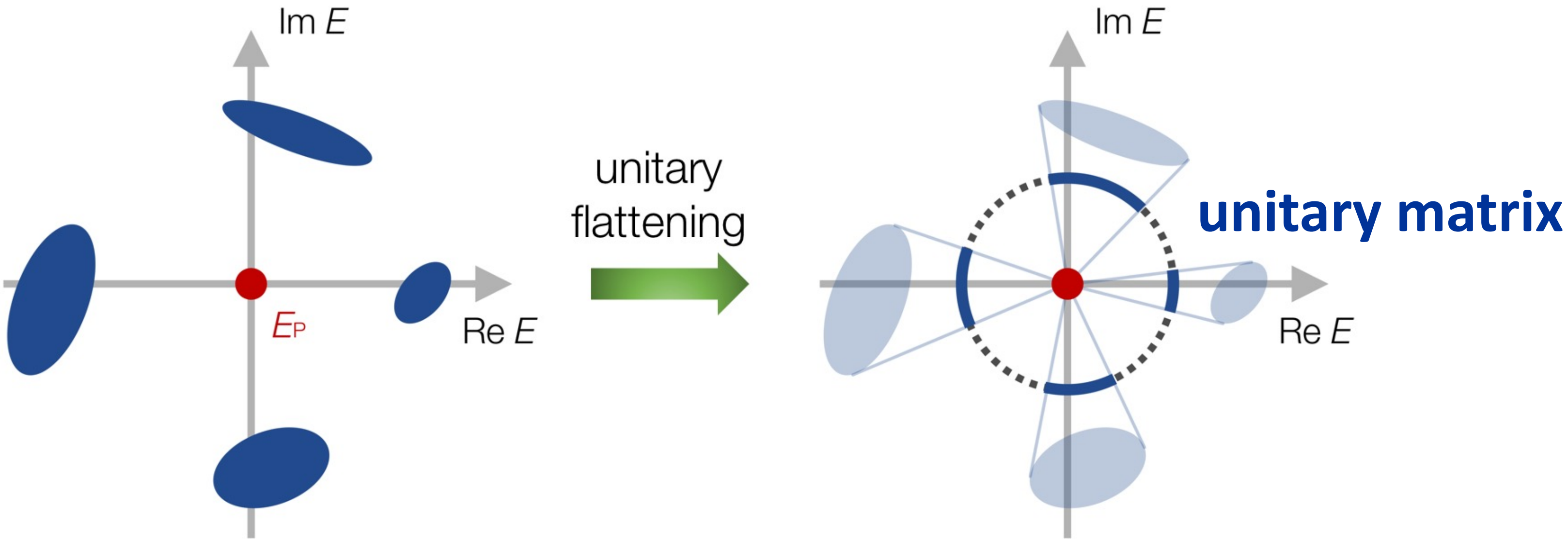
Point gap: unitary flattening

Unitary flattening for a point gap:

A non-Hermitian Hamiltonian with a **point gap** can be continuously deformed into a **unitary matrix** while having symmetry and the point gap.

non-Hermitian H ⟷ unitary U ⟷ **Hermitian** $\begin{pmatrix} 0 & U \\ U^\dagger & 0 \end{pmatrix}$

Classification is well established!



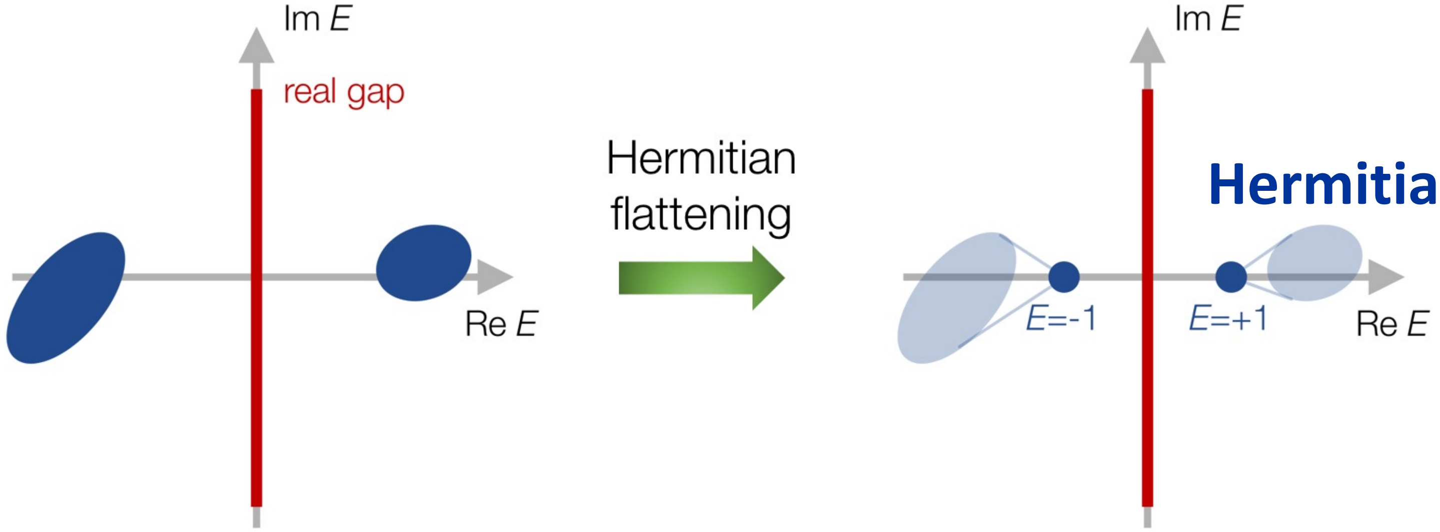
Line gap: Hermitian flattening

Hermitian flattening for a line gap:

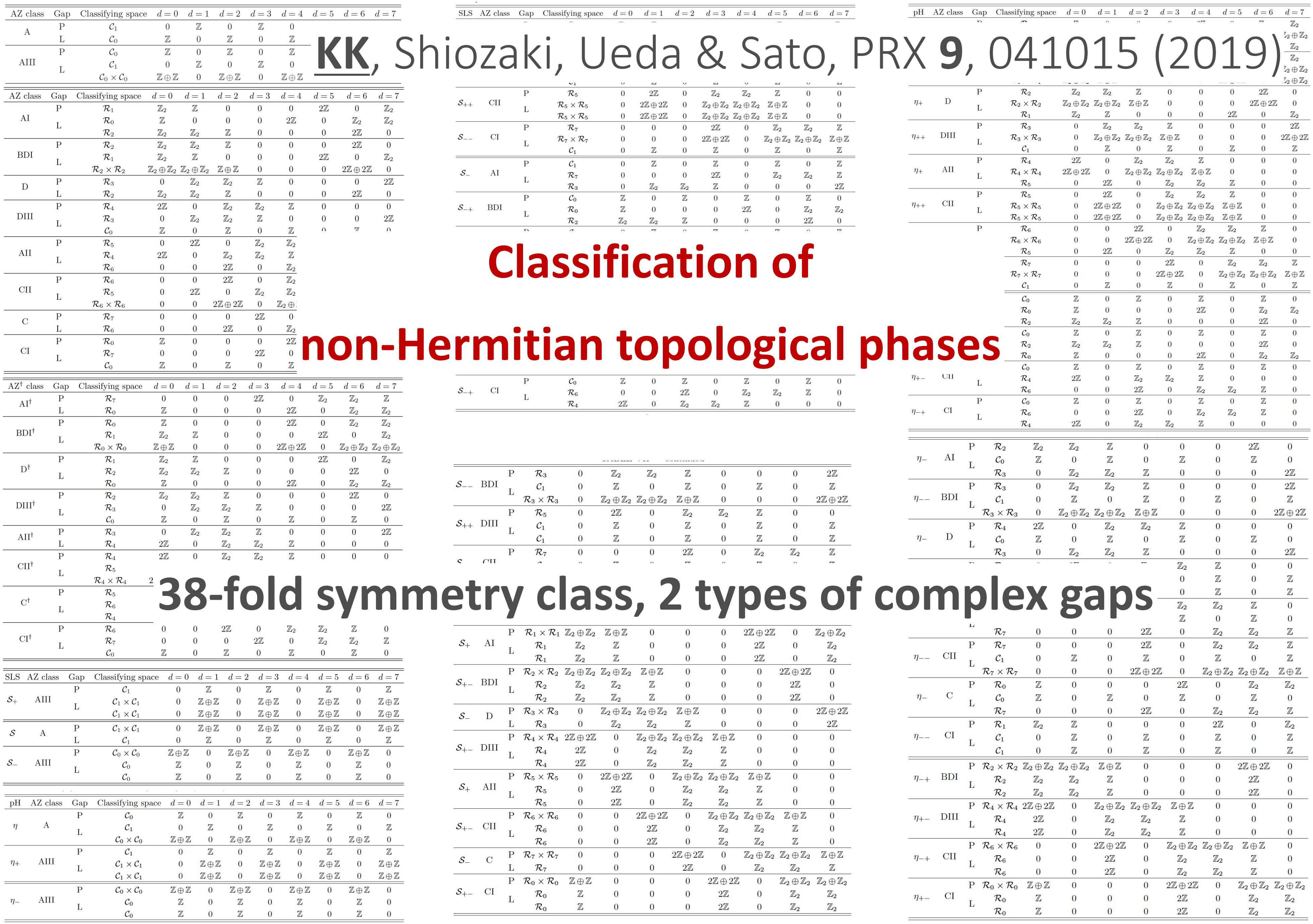
A non-Hermitian Hamiltonian with a **line gap** can be continuously deformed into a **Hermitian matrix** while having symmetry and the line gap.

non-Hermitian H $\longleftrightarrow$ **Hermitian** $\tilde{H}$

Classification is well established!



Hermitian matrix



KK, Shiozaki, Ueda & Sato, PRX **9**, 041015 (2019)

Classification of non-Hermitian topological phases

38-fold symmetry class, 2 types of complex gaps

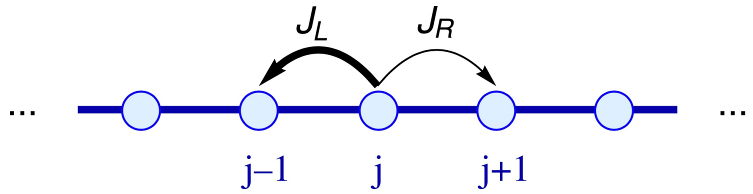
Intrinsic non-Hermitian topology

Line-gap topology: stability of Hermitian topology against non-Hermiticity

⟷ **Point-gap topology: intrinsic non-Hermitian topology**

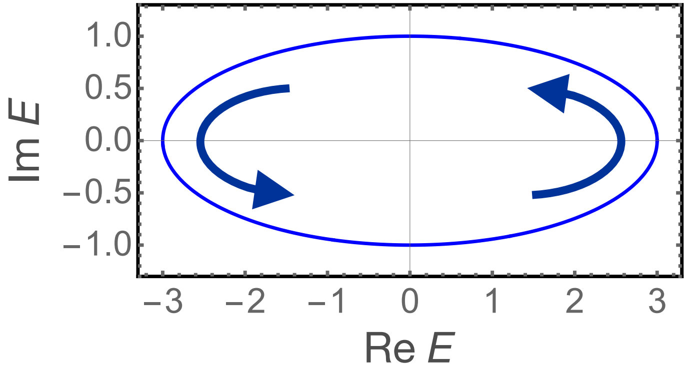
- **Hatano-Nelson model** Hatano & Nelson, PRL **77**, 570 (1996)

asymmetric hopping



$$\hat{H}_{\mathrm{HN}} = \sum_i \left(J_{\mathrm{R}} \hat{c}_{i+1}^{\dagger} \hat{c}_i + J_{\mathrm{L}} \hat{c}_i^{\dagger} \hat{c}_{i+1} \right)$$

$$H_{\mathrm{HN}}(k) = J_{\mathrm{R}} e^{\mathrm{i}k} + J_{\mathrm{L}} e^{-\mathrm{i}k}$$



Winding of complex energy!

$$W(E) := \oint \frac{dk}{2\pi \mathrm{i}} \frac{d}{dk} \log \det \left(H(k) - E \right)$$

(ill-defined in Hermitian systems)

Gong, Ashida, **KK** *et al.*, PRX **8**, 031079 (2018)

☆ Physical consequences of intrinsic non-Hermitian topology?

bulk-boundary correspondence, topological field theory, ……

Topological Origin of Non-Hermitian Skin Effects

Okuma, **Kawabata**, Shiozaki & Sato, PRL **124**, 086801 (2020)

Kawabata, Okuma & Sato, PRB **101**, 195147 (2020)

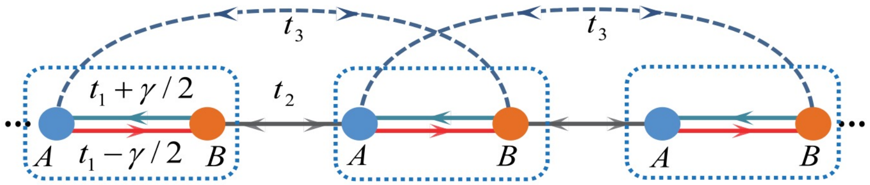
Kawabata, Sato & Shiozaki, PRB **102**, 205118 (2020)

Bulk-boundary correspondence (BBC)

Emergence of boundary states due to nontrivial bulk topology.

→ **Non-Hermiticity alters the nature of BBC!**

e.g. Non-Hermitian (non-reciprocal) SSH model



$$H_{\mathrm{SSH}}(k) = \begin{pmatrix} 0 & t_1 \underline{+\gamma} + t_2 e^{-ik} \\ t_1 \underline{-\gamma} + t_2 e^{ik} & 0 \end{pmatrix}$$

asymmetric hopping

Sublattice symmetry: $\sigma_z H_{\mathrm{SSH}}(k)\,\sigma_z = -H(k)$

→ winding number (integer, Z)

☆ Zero modes appear, but are localized only at one edge.

Lee, PRL **116,** 133903 (2016); Yao & Wang, PRL **121**, 086803 (2018)

Non-Hermitian skin effect

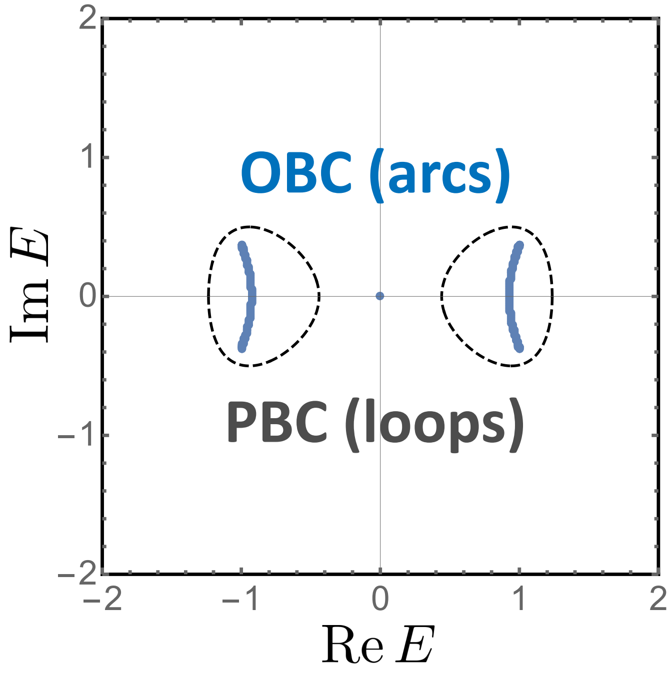
Non-Hermitian skin effect

All the eigenstates are localized at the edges!

The bulk Hamiltonian strongly depends on the boundary conditions!

$$H^{(\mathrm{PBC})}(k) \neq H^{(\mathrm{OBC})}(k)$$

$$H_{\mathrm{SSH}}^{(\mathrm{OBC})}(k) = \begin{pmatrix} 0 & \sqrt{t_1^2 - \gamma^2} + t_2 e^{-\mathrm{i}k} \\ \sqrt{t_1^2 - \gamma^2} + t_2 e^{\mathrm{i}k} & 0 \end{pmatrix} \neq H_{\mathrm{SSH}}^{(\mathrm{PBC})}(k)$$



Modified bulk-boundary correspondence

Zero modes are predicted by $H^{(\mathrm{OBC})}(k)$ rather than $H^{(\mathrm{PBC})}(k)$

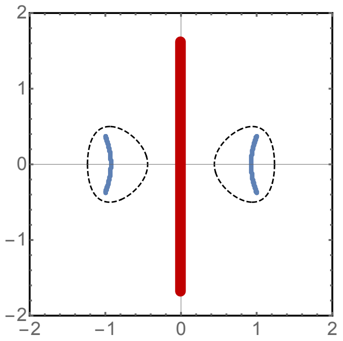
Yao & Wang, PRL **121**, 086803 (2018); Kunst *et al.*, PRL **121**, 026808 (2018)

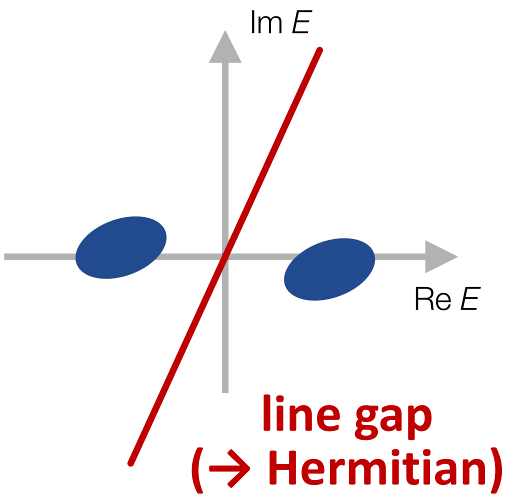
BBC in non-Hermitian systems

The modified BBC is developed for **line-gap topology**.

Non-Hermitian Hamiltonian with a line gap

- → **Hermitian** Hamiltonian
- → The modified BBC





line gap
(→ Hermitian)

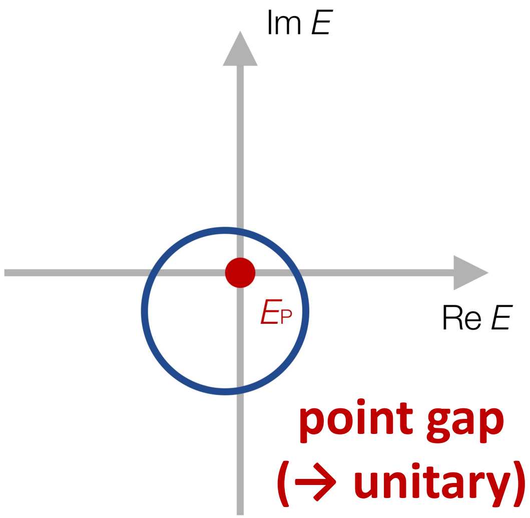
☆ BBC for point-gap topology?

Non-Hermitian Hamiltonian with a point gap

- → **Unitary** Hamiltonian
- → Not necessarily deformed into a Hermitian one

  Intrinsic to non-Hermitian Hamiltonians

  NO modified BBC

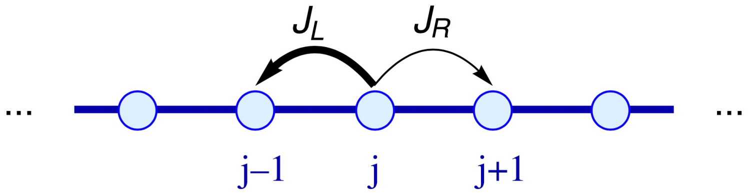


point gap
(→ unitary)

Hatano-Nelson model

- **Hatano-Nelson model**

asymmetric hopping



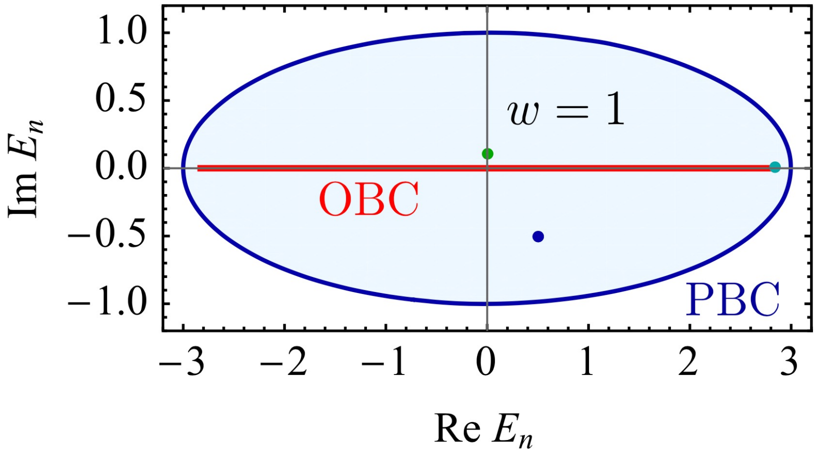
Hatano & Nelson, PRL **77**, 570 (1996)

$$\hat{H}_{\mathrm{HN}} = \sum_{i} \left(J_{\mathrm{R}} \hat{c}_{i+1}^{\dagger} \hat{c}_{i} + J_{\mathrm{L}} \hat{c}_{i}^{\dagger} \hat{c}_{i+1} \right)$$

$$H_{\mathrm{HN}} \left(k \right) = J_{\mathrm{R}} e^{\mathrm{i}k} + J_{\mathrm{L}} e^{-\mathrm{i}k}$$

Topological invariant (winding #)

$$W \left(E \right) := \oint \frac{dk}{2\pi \mathrm{i}} \frac{d}{dk} \log \det \left(H \left(k \right) - E \right)$$



Gong, Ashida, **KK** *et al.*,
PRX **8**, 031079 (2018)

- Under PBC, $W \left(E \right) = \pm 1$
- Under OBC, however,

the skin effect occurs

no point gap is open

Skin effect as point-gap topology

Is it possible to have $H^{(\mathrm{OBC})}$ with point-gap topology?

→ **No! $H^{(\mathrm{OBC})}$ is always topologically trivial for a point gap.**

(can be nontrivial for a line gap)

Then, what does point-gap topology imply?

→ **Point-gap topology leads to the non-Hermitian skin effect!**

(intrinsic NH topology) (intrinsic NH phenomenon)

$W(E) \neq 0$: skin effect

$W(E) = 0$: no skin effect

Okuma, **KK**, Shiozaki & Sato, PRL **124**, 086801 (2020)

cf. K. Zhang, Z. Yang & C. Fang, PRL **125**, 126402 (2020).

Same conclusion, but different approach

(BBC for intrinsic non-Hermitian topology)

Skin effect — **Point-gap topology**

Classification?

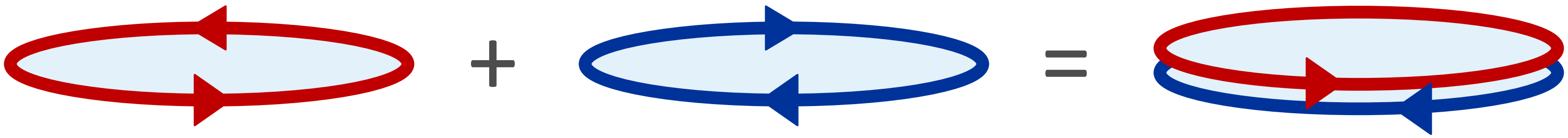
Classification

Symmetry-protected skin effects?

Symmetry-protected topological phases

Z_2 skin effect protected by TRS (1)

- **Z_2 topological phase in Hermitian systems**



Haldane model

Z invariant (Chern #)

chiral edge states

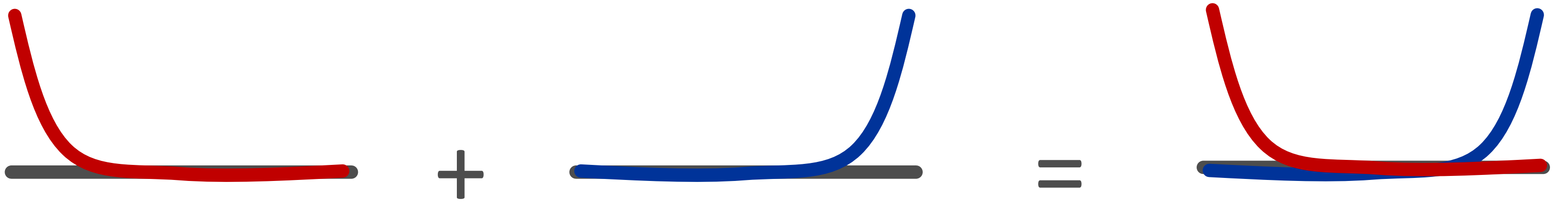
time-reversed partner

Kane-Mele model

Z_2 invariant with TRS

helical edge states

- **Z_2 skin effect in non-Hermitian systems**



localized at left (+W)

Hatano-Nelson model

Z invariant (point gap)

localized at right (-W)

time-reversed partner

(reciprocal)

localized at both!

Reciprocal skin effect with the Z_2 invariant protected by time-reversal symmetry

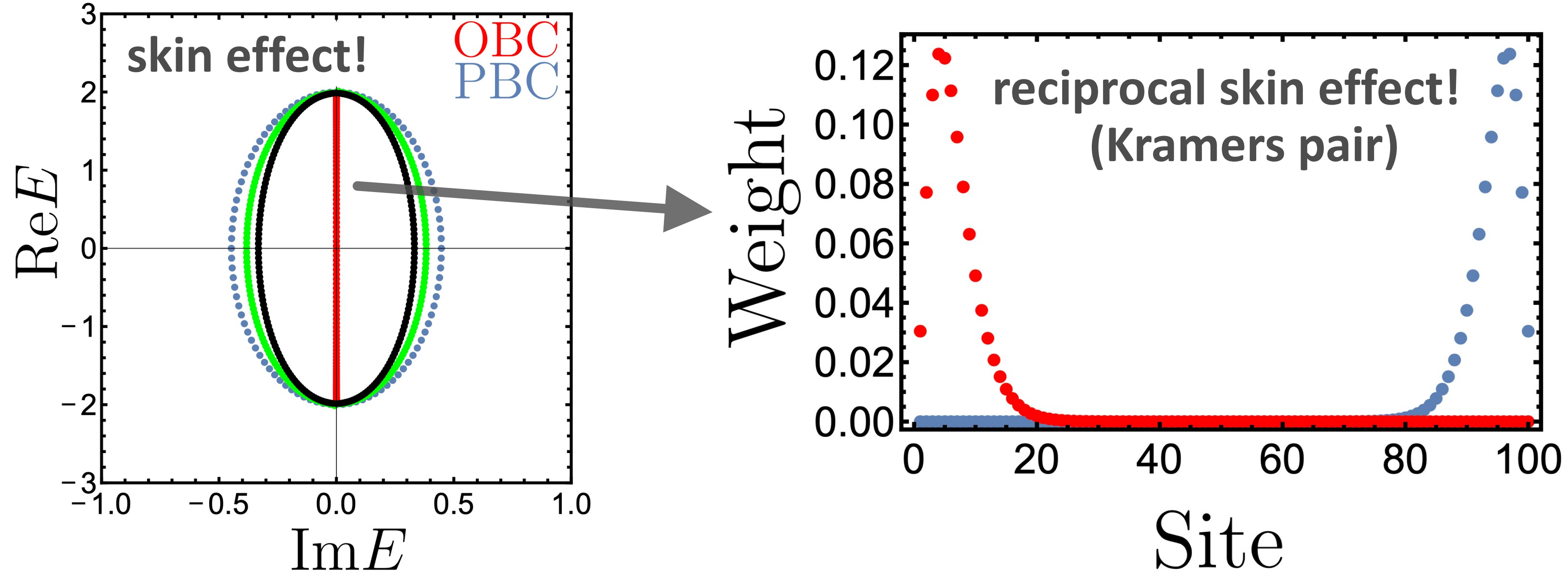
Z_2 skin effect protected by TRS (2)

$$H\left(k\right)=\begin{pmatrix} H_{\mathrm{HN}}\left(k\right) & 2\Delta\sin k \\ \underline{2\Delta\sin k} & H_{\mathrm{HN}}^{T}\left(-k\right) \end{pmatrix}=2t\cos k+2\Delta\left(\sin k\right)\sigma_x+2\mathrm{i}g\left(\sin k\right)\sigma_z,$$

(symmetry-preserving perturbation)

$$\mathrm{TRS}^{\dagger}:\;\left(\mathrm{i}\sigma_y\right)H^{T}\left(k\right)\left(\mathrm{i}\sigma_y\right)^{-1}=H\left(-k\right),\quad\left(\mathrm{i}\sigma_y\right)\left(\mathrm{i}\sigma_y\right)^{*}=-1$$

→ Z_2 topological, Kramers degeneracy


skin effect!

reciprocal skin effect!
(Kramers pair)

Topological Field Theory of Non-Hermitian Systems

Kawabata, Shiozaki & Ryu, PRL **126**, 216405 (2021)

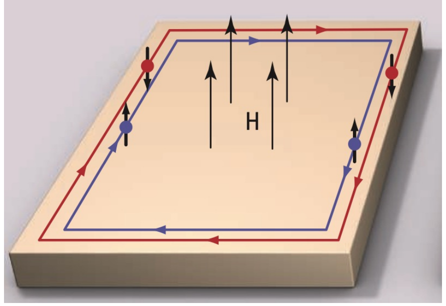
Topological field theory

☆ Topological phenomena are universally described by field theory.

- **(2+1)-D Chern-Simons theory: quantum Hall effect**

$$S\left[\boldsymbol{A}\right]=\frac{C_1}{4\pi}\int d^2x\int dt\,\varepsilon^{\mu\nu\tau}A_\mu\partial_\nu A_\tau$$

$$\longrightarrow\quad j^i=\frac{\delta S}{\delta A_i}=\frac{C_1}{2\pi}\varepsilon^{ij}E_j$$



Oh, Science **340**, 153 (2013)

$S\left[\boldsymbol{A}\right]$ is **gauge dependent** in the presence of a boundary (**anomaly**)

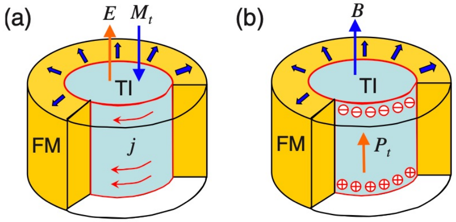
⟶ Compensated by an anomaly at the boundary, i.e., **chiral edge states**.

(field-theoretic understanding of bulk-boundary correspondence)

- **Axion electrodynamics for (3+1)-D topological insulators**

$$S=\frac{\theta}{4\pi^2}\int d^3x\int dt\,\boldsymbol{B}\cdot\boldsymbol{E}$$

Qi, Hughes & Zhang, PRB **78**, 195424 (2008)



Derivation of topological field theory

The topological action S is derived from a microscopic Hamiltonian H

$$Z\left[\underline{\boldsymbol{A},\phi}\right]=\int\mathcal{D}\bar{\psi}\mathcal{D}\psi\,e^{\mathrm{i}\mathcal{S}},\quad\mathcal{S}=\int d^dx\int dt\,\bar{\psi}\left[\mathrm{i}\partial_t+\phi-H\left(-\mathrm{i}\partial_x-\boldsymbol{A}\right)\right]\underline{\psi}$$

(gauge potential)

(matter)

Let us integrate out the matter degrees of freedom (i.e., ψ)

$$Z\left[\boldsymbol{A},\phi\right]=\det\left[\mathrm{i}\omega+\phi-H\left(\boldsymbol{k}-\boldsymbol{A}\right)\right]$$

effective action: $e^{\mathrm{i}S[\boldsymbol{A},\phi]}:=Z\left[\boldsymbol{A},\phi\right]/Z\left[0\right]$

The topological invariant is given by the **Green's function**:

$$G_0^{-1}\left(\boldsymbol{k},\omega\right):=\mathrm{i}\omega-H\left(\boldsymbol{k}\right)$$

e.g. Chern insulator $H\left(\boldsymbol{k}\right)=k_x\sigma_x+k_y\sigma_y+m\sigma_z$

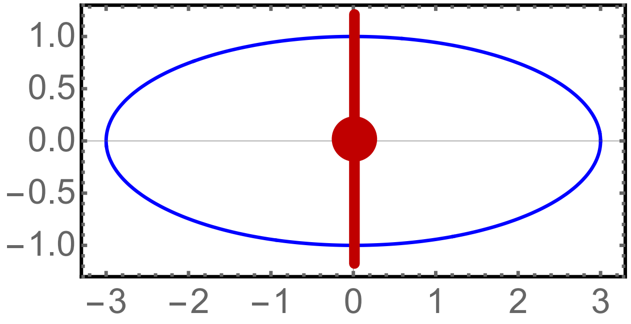
→ (2+1)-D Chern-Simons theory

☆ **An energy gap ensures the well-defined topological action**

Breakdown of the spacetime formulation

Can we have a topological action from a non-Hermitian Hamiltonian?
(e.g., Hatano-Nelson model)

→ **No, we do not have a well-defined action!**



Point gap: open (gapped)

Line gap: closed (gapless)

→ Divergence of the action

(The system looks like a metal rather than an insulator)

The above formulation assumes the Gibbs state for equilibrium, and focuses on the ground state.

$$Z = \mathrm{tr}\, e^{-\beta \hat{H}}$$

→ **Intrinsic non-Hermitian topology: out of equilibrium**

The Gibbs state and the ground state are no longer relevant……

Space formulation

“Time” should play a special role out of equilibrium

Let us Fourier transform $\psi\left(\boldsymbol{x},t\right)=\int\psi_E\left(\boldsymbol{x}\right)e^{-\mathrm{i}Et}dE$

→ **Spatial field theory**

KK, Shiozaki & Ryu, PRL **126**, 216305 (2021)

$$Z_E\left[\boldsymbol{A}\right]=\int\mathcal{D}\bar{\psi}_E\mathcal{D}\psi_E\,e^{\mathrm{i}\mathcal{S}_E},\quad\mathcal{S}_E=\int d^{\mathrm{d}}x\,\bar{\psi}_E\left[H\left(\boldsymbol{k}-\boldsymbol{A}\right)-E\right]\psi_E$$

generating function of the Green’s function $\left(E-H\left(\boldsymbol{k}\right)\right)^{-1}$
captures all physical information!

☆ **We have a well-defined action for intrinsic non-Hermitian topology!**

$$G_0^{-1}\left(\boldsymbol{k},\omega\right):=\mathrm{i}\omega-H_{\mathrm{H}}\left(\boldsymbol{k}\right)\quad\Longleftrightarrow\quad\mathrm{i}k_{d+1}-H_{\mathrm{H}}\left(\boldsymbol{k}\right)=H_{\mathrm{NH}}\left(\boldsymbol{k},k_{d+1}\right)$$

(Green’s function)

$\omega\leftrightarrow k_{d+1}$
$t\leftrightarrow x_{d+1}$

[(*d*-1)+1]-D Hermitian systems — **(*d*+0)-D Non-Hermitian systems**

One dimension

For $H(k)$ in 1D (such as the Hatano-Nelson model),

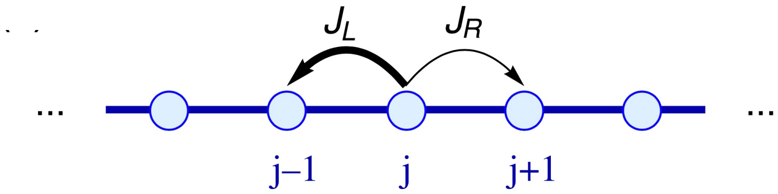
$$S_E[A] = \mathrm{i}\,\mathrm{tr}\left[(H-E)A\right] = W_1(E)\int A(x)\,dx$$

(1+0)-D Chern-Simons theory!

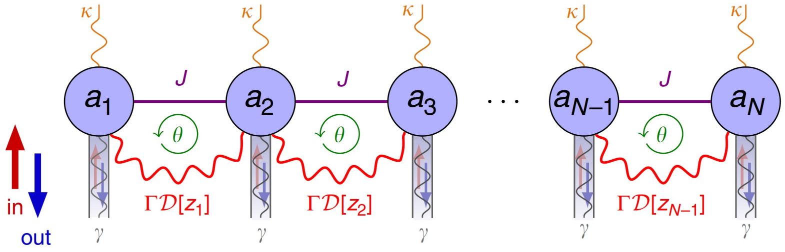
cf. (0+1)-D CS theory for 0D Hermitian systems $S[A] = C_0\int \phi(t)\,dt$

- Current: $j = \dfrac{\delta S_E}{\delta A} = W_1$

Nonreciprocal transport due to non-Hermiticity!



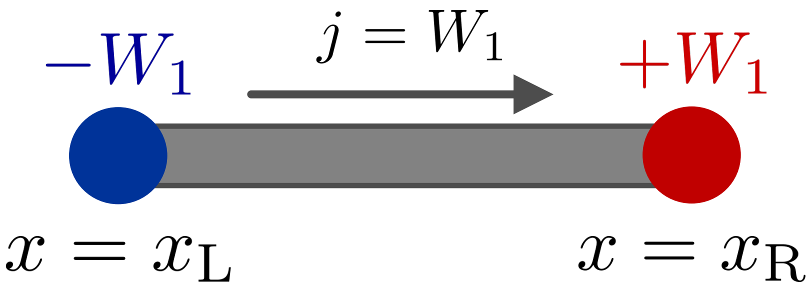
Application: directional amplification (laser)



Wanjura *et al.*, Nat. Commun. **11**, 3149 (2020)

Skin effect as an anomaly

1D non-Hermitian systems: $S_E[A] = W_1(E) \int A(x)\, dx$



Gauge transformation: $A \to A + df/dx$

The action transforms as

$$S_E \to S_E + W_1(E)\left[f(x_{\mathrm{R}}) - f(x_{\mathrm{L}})\right]$$

NOT gauge invariant! (anomaly)

To retain gauge invariance, additional system is needed at the boundary:

$$S_E^{\text{boundary}} = -W_1(E)\left[\underline{\varphi(x_{\mathrm{R}})} - \underline{\varphi(x_{\mathrm{L}})}\right]$$

(phase of the wavefunction)

$$\longrightarrow S_E^{\text{boundary}} \to S_E^{\text{boundary}} - W_1(E)\left[f(x_{\mathrm{R}}) - f(x_{\mathrm{L}})\right]$$

$S_E + S_E^{\text{boundary}}$ **is gauge invariant**

S_E^{boundary} **describes a pair of charges at the boundary: skin modes**

☆ The non-Hermitian skin effect is a signature of an anomaly!

Three dimensions: chiral magnetic effect

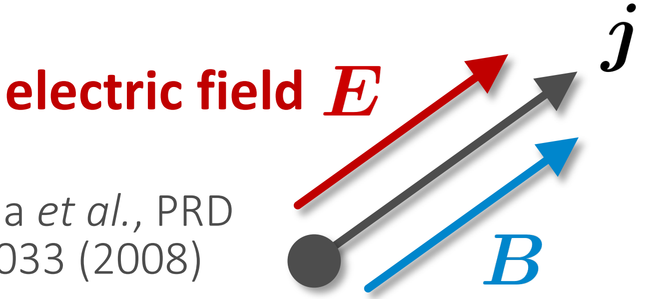
Generally, *d*-dimensional non-Hermitian systems are described by the (*d*+0)-dimensional Chern-Simons theory (*d*: odd)

In three dimensions, we have the **(3+0)-D Chern-Simons theory**:

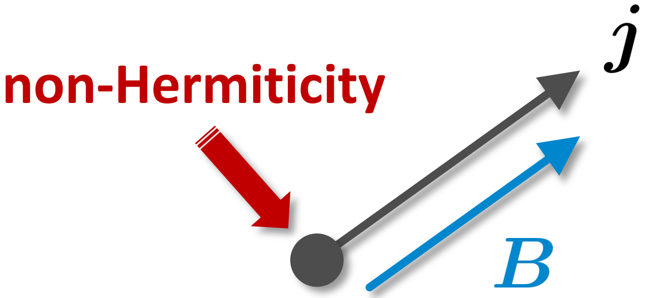
$$S_E\left[\boldsymbol{A}\right] = \frac{W_3\left(E\right)}{4\pi} \int \varepsilon^{ijk} A_i \partial_j A_k \, d^3x$$

cf. (2+1)-D CS theory for 2D Hermitian systems (QHE)

- Current: $\boldsymbol{j} = \frac{W_3}{2\pi}\boldsymbol{B}$ **chiral magnetic effect!**



Fukushima *et al.*, PRD **78**, 074033 (2008)



- Application: directional laser controlled by a magnetic field

cf. lattice realization: Bessho & Sato, PRL **127**, 196404 (2021)

Chiral magnetic skin effect

- (2+1)-D Chern-Simons theory (quantum Hall effect)

Boundary exhibits a **chiral anomaly**: $\partial_x j_x^A + \partial_t j_t^A = E/\pi$

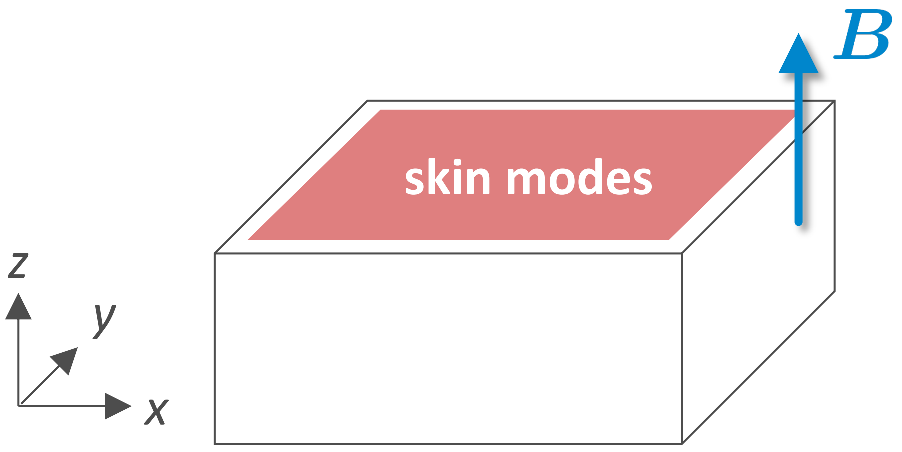
(axial current)

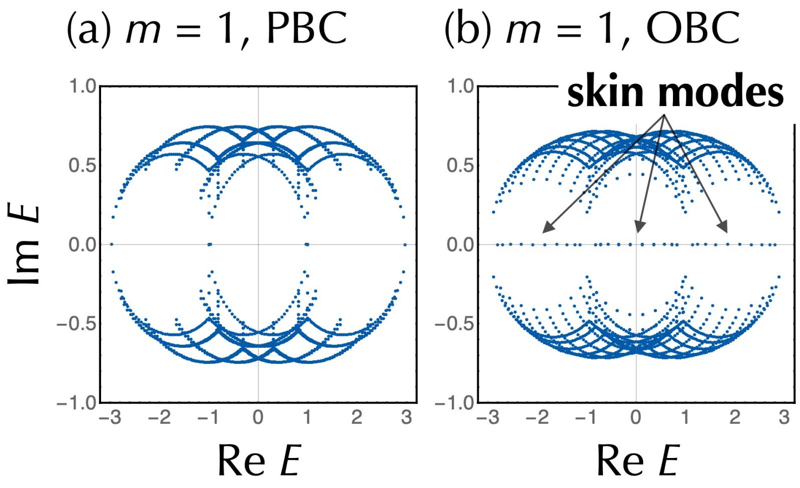
- (3+0)-D Chern-Simons theory for 3D non-Hermitian systems

Boundary also exhibits a **chiral anomaly (*t* → *y*, *E* → *B*)**:

$$\nabla \cdot \boldsymbol{j}^A = B/\pi \quad \longleftrightarrow \quad N_\pm = \pm\Phi/2\pi$$

The number of the skin modes is given by the number of the fluxes





Summary

- 38-fold internal symmetry in non-Hermitian physics.
- Non-Hermiticity gives rise to unique topological phases without Hermitian analogs.
- Intrinsic non-Hermitian topology leads to the skin effects.
- Spatial field theory for intrinsic non-Hermitian topology; the skin effect is a signature of a non-Hermitian anomaly.