東京大学
THE UNIVERSITY OF TOKYO

Nonunitary Scaling Theory of Non-Hermitian Localization

Kohei Kawabata (University of Tokyo)

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Collaborator: Shinsei Ryu (Chicago → Princeton)

arXiv: 2005.00604

Outline

1. Introduction and motivation

- Non-Hermitian localization

2. Scaling theory of non-Hermitian localization

- Nonunitary scattering matrices
- Scaling equations
- Two-parameter scaling

3. Symmetry and non-Hermitian localization

- No delocalization due to reciprocity (class $\mathrm{AI}^{\dagger}$)
- Bidirectional delocalization protected by symplectic reciprocity (class $\mathrm{AII}^{\dagger}$)

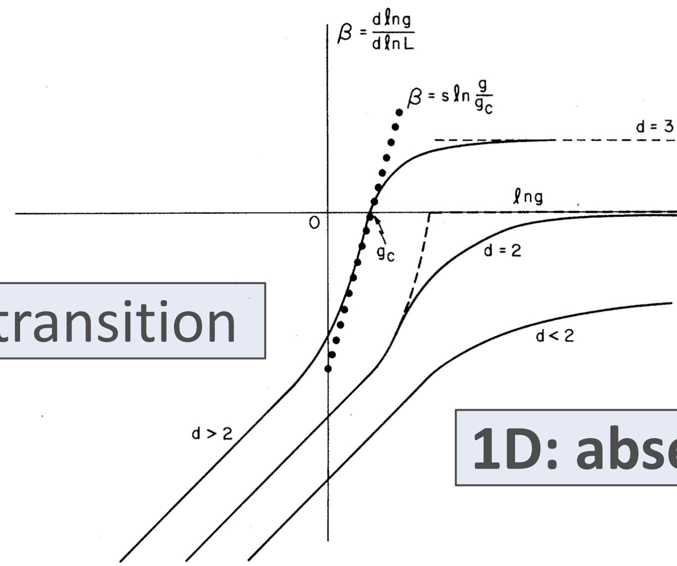
Scaling theory of localization

Anderson localization: disorder-induced localization of coherent waves

Anderson, PR **109**, 1492 (1958)

Unified understanding of localization: **scaling theory**

Abrahams *et al.*, PRL **42**, 673 (1979)



3D: localization transition

1D: absence of delocalization

Assumption of the scaling theory: **one-parameter scaling**

$$\beta(G) := \frac{d\log G}{d\log L}$$ is given solely by G

G : conductance L : length scale

Non-Hermitian physics

Despite the enormous success, Anderson localization has been mainly investigated in **closed and conservative (Hermitian)** systems.

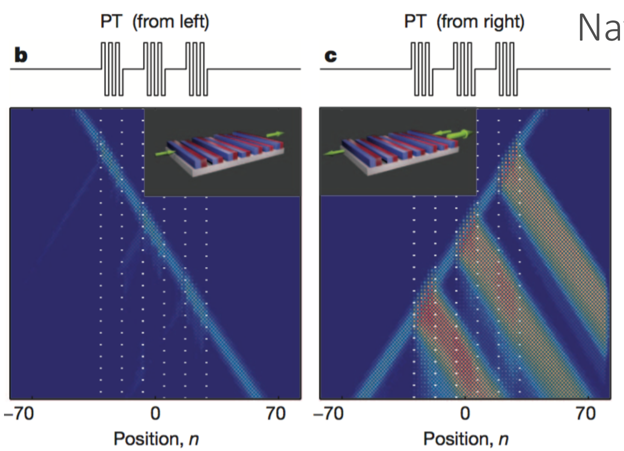
→ **Richer properties appear in non-Hermitian systems.**
(non-Hermiticity can induce decoherence)

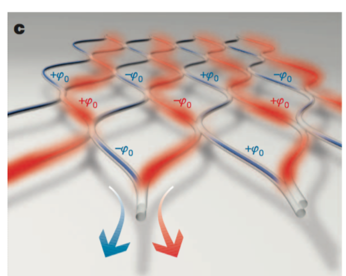
☆ Non-Hermiticity arises from **non-conservation of energy and/or particles** (e.g. non-equilibrium open systems).

- **Photonic lattices with gain/loss**

Unidirectional light transport.

Regensburger *et al.*, Nature **488**, 167 (2012)

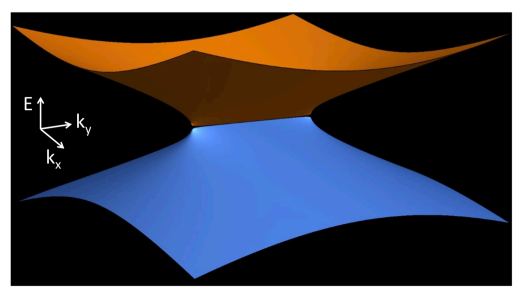


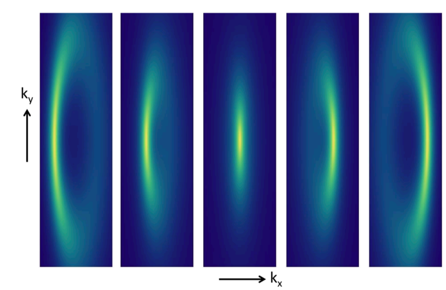


- **Finite-lifetime quasiparticles**

Bulk Fermi arc due to non-Hermitian self-energy.

Kozii & Fu, arXiv: 1708.05841





Hatano-Nelson model

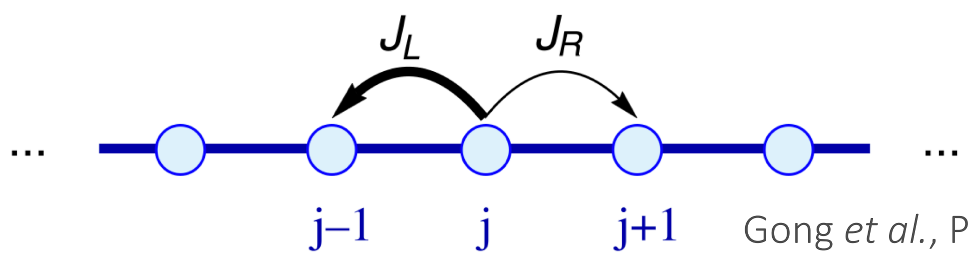
Delocalization is impossible in Hermitian systems in 1D.

→ **Delocalization is possible in the presence of non-Hermiticity!**

Hatano & Nelson, PRL **77**, 570 (1996)

$$\hat{H} = \sum_n \left\{ -\frac{1}{2} \left[\underline{\left(J + \frac{\gamma}{2} \right)} \hat{c}^\dagger_{n+1} \hat{c}_n + \underline{\left(J - \frac{\gamma}{2} \right)} \hat{c}^\dagger_n \hat{c}_{n+1} \right] + m_n \hat{c}^\dagger_n \hat{c}_n \right\}$$

asymmetric hopping



Gong *et al.*, PRX **8**, 031079 (2018)

Strong non-Hermiticity leads to delocalization even in 1D.

Many subsequent works:

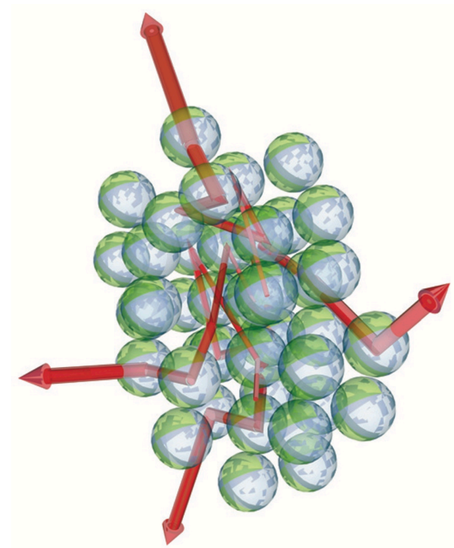
Efetov, PRL **79**, 491 (1997); Feinberg *et al.*, Nucl. Phys. B **504**, 579 (1997); Brouwer *et al.*, PRB **56**, R4333(R) (1997); Goldsheid *et al.*, PRL **80**, 2897 (1998); Mudry *et al.*, PRL **80**, 4257 (1998); Yurkevich *et al.*, PRL **82**, 5080 (1999)

Random laser

Random lasers can be described by non-Hermitian disordered systems:

$$\hat{H} = \sum_n \left\{ -\frac{J}{2}\left(\hat{c}^\dagger_{n+1}\hat{c}_n + \hat{c}^\dagger_n\hat{c}_{n+1}\right) + \underline{(m_n + \mathrm{i}\gamma_n)}\, \hat{c}^\dagger_n\hat{c}_n \right\}$$

complex potential (gain and/or loss)



Wiersma, Nat. Phys. **4**, 359 (2008)

No delocalization occurs in 1D and 2D, in contrast to the Hatano-Nelson model.

Delocalization occurs in 3D, and its criticality seems to be the **same as the Hermitian Anderson model.**

e.g. Tzortzakakis *et al.*, PRB **101**, 014202 (2020); Huang *et al.*, PRB **101**, 014204 (2020) (subsequent talk)

Motivation

・Non-Hermitian disordered systems can exhibit delocalization **even in 1D** (e.g. Hatano-Nelson model).

・For Hermitian systems, **scaling theory** forbids delocalization in 1D.

→ **Scaling theory of localization should be changed in non-Hermitian systems.**

・While the Hatano-Nelson model exhibits delocalization, random lasers do not exhibit delocalization.

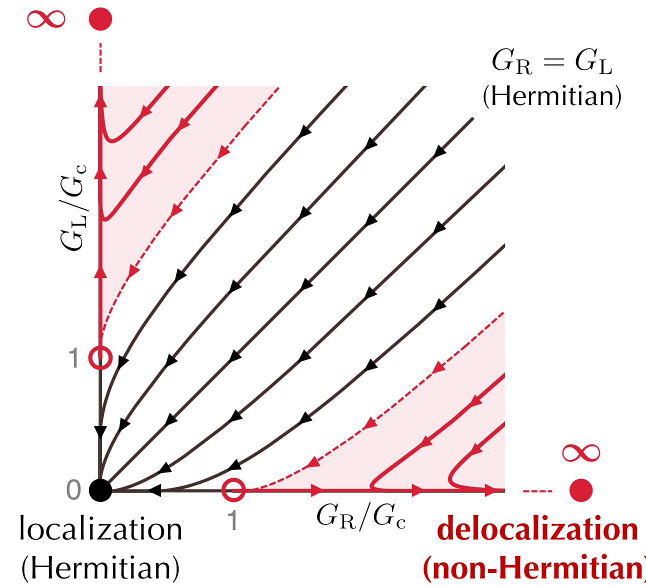
→ **What determines the universality of non-Hermitian localization?**

Results

- We develop a scaling theory of localization in non-Hermitian systems.

- Non-Hermiticity introduces a new scale and **breaks down the one-parameter scaling**.

→ **Two-parameter scaling**
(origin of delocalization in 1D)



- **Threefold universality based on reciprocity**

Class	Symmetry	Delocalization	Conductances
A	No	Unidirectional	$e^{(\pm\gamma-1/\ell)L}$
AI†	$H^T = H$	No	$e^{-L/\ell}$
AII†	$\sigma_2 H^T \sigma_2^{-1} = H$	Bidirectional	$e^{(\lvert\gamma\rvert-1/\ell)L}$

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Nonunitary scattering matrices

The conductance is obtained by the **scattering matrix** (Landauer formula)

(reflection from right to left) (transmission from right to left)

$$S = \begin{pmatrix} r_{\mathrm{L}} & t_{\mathrm{L}} \\ t_{\mathrm{R}} & r_{\mathrm{R}} \end{pmatrix}$$

(transmission from left to right) (reflection from left to right)

Hermiticity of the Hamiltonian H

⟷ **Unitarity of the scattering matrix** S

☆ The conductance is not necessarily finite in non-Hermitian systems.
(direct consequence of non-conservation of particles)

Unitarity of S ⟷ $S^\dagger S = 1$

⟶ $T + R = 1$ ⟶ $T \leq 1$

(transmission and reflection amplitudes)

Nonunitary regime

Scaling function $\beta(G) := \frac{d \log G}{d \log L}$

G : conductance

L : length scale

Abrahams *et al.*, PRL **42**, 673 (1979)

Localized regime

localization → $G \propto e^{-L/\xi}$ → $\beta = \log G < 0$

Diffusive regime

Ohm's law → $G \propto L^{d-2}$ → $\beta = d - 2$

$$\begin{cases} d = 1 & \beta < 0 \\ d = 3 & \beta > 0 \end{cases}$$

- $d = 1$, $\beta < 0$: **no delocalization**
- $d = 3$, $\beta > 0$: **delocalization (transition)**

Non-unitary regime

non-conservation of particles → (amplification) $G \propto e^{\gamma L}$ → $\beta = \log G \underline{> 0}$

☆ **Non-unitarity enables delocalization even in 1D.**

Thouless criterion

Thouless, Phys. Rep. **13**, 93 (1974)

☆ **Thouless time**: time for a particle to reach one end from the other end

diffusive transport ⟶ $t_{\mathrm{Th}} \propto L^2$

To realize this diffusive transport, we need $t_{\mathrm{Th}} < \Delta t$

$$\Delta t \propto (\Delta E)^{-1} \propto L^d$$

(energy-level spacing)

Diffusion is impossible in 1D: $d = 1 \to t_{\mathrm{Th}}/\Delta t \propto L$

☆ **Non-unitarity enables a different type of transport.**

particle inflow from the environment > localization

⟶ **ballistic transport** $t_{\mathrm{N}} \propto L$

⟶ $t_{\mathrm{N}}/\Delta t \propto L^{1-d}$

can be smaller even in 1D

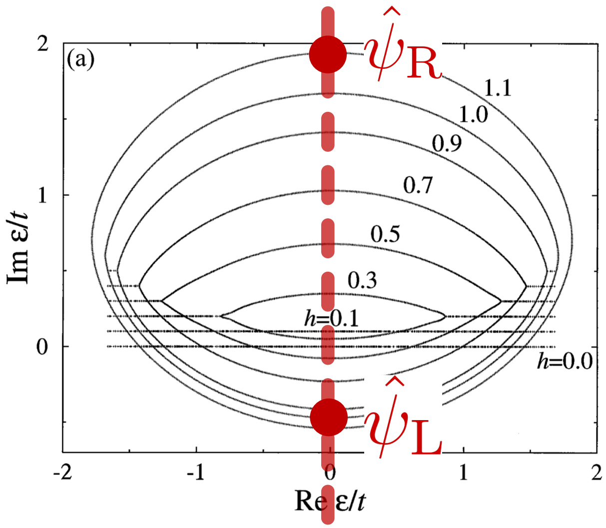
R. Hamazaki (private communication)

Continuum model

Hatano-Nelson model

$$\hat{H} = \sum_n \left\{ -\frac{1}{2} \left[\left(J + \frac{\gamma}{2} \right) \hat{c}_{n+1}^\dagger \hat{c}_n + \left(J - \frac{\gamma}{2} \right) \hat{c}_n^\dagger \hat{c}_{n+1} \right] + m_n \hat{c}_n^\dagger \hat{c}_n \right\}$$

To understand a universal feature of non-Hermitian localization, we consider a **continuum model** of the Hatano-Nelson model.



$\mathrm{Re}\, E = 0$
("Fermi surface")

Hatano & Nelson, PRL **77**, 570 (1996)

$$\hat{c}_n = e^{\mathrm{i}k_{\rm F}n} \underline{\hat{\psi}_{\rm R}(n)} + e^{-\mathrm{i}k_{\rm F}n} \underline{\hat{\psi}_{\rm L}(n)}$$

right-moving　　left-moving

$$\longrightarrow \quad \hat{H} = \int dx \, (\hat{\psi}_{\rm R}^\dagger \; \hat{\psi}_{\rm L}^\dagger) \, h_{\rm A} \, (\hat{\psi}_{\rm R} \; \hat{\psi}_{\rm L})^T$$

$$h_{\rm A} = (-\mathrm{i}\partial_x + \mathrm{i}\gamma/2)\, \tau_3 + \underline{m_0(x)} + \underline{m_1(x)}\, \tau_1$$

disorder (Gaussian)

☆ Independent from specific details of the model

Scaling equations

We obtain the scaling equations for the continuum model h_{A}
(non-unitary generalization of the DMPK equations)

Dorokhov, JETP Lett. **36**, 318 (1982);
Mello *et al.*, Ann. Phys. **181**, 290 (1988)

$$\frac{\langle dT_{\mathrm{R/L}} \rangle}{dL} = \pm \underline{\gamma} T_{\mathrm{R/L}} - \frac{T_{\mathrm{R/L}} \left(1 - R_{\mathrm{L/R}} \right)}{\underline{\ell}}$$

non-Hermiticity

mean-free path

- For $\gamma > 0$, T_{R} is amplified, and T_{L} is attenuated $(T_{\mathrm{R}} \neq T_{\mathrm{L}})$.
- $T_{\mathrm{R/L}} =: e^{\pm \gamma L} \underline{\tilde{T}}$

transmission amplitude in Hermitian systems

$$\tilde{T} \sim e^{-L/\ell} \quad (L \gg \ell)$$

$$\longrightarrow \begin{cases} G_{\mathrm{R}} = e^{+\gamma L} \tilde{G} = e^{(\gamma - 1/\ell) L} \\ G_{\mathrm{L}} = e^{-\gamma L} \tilde{G} = e^{(-\gamma - 1/\ell) L} \end{cases}$$

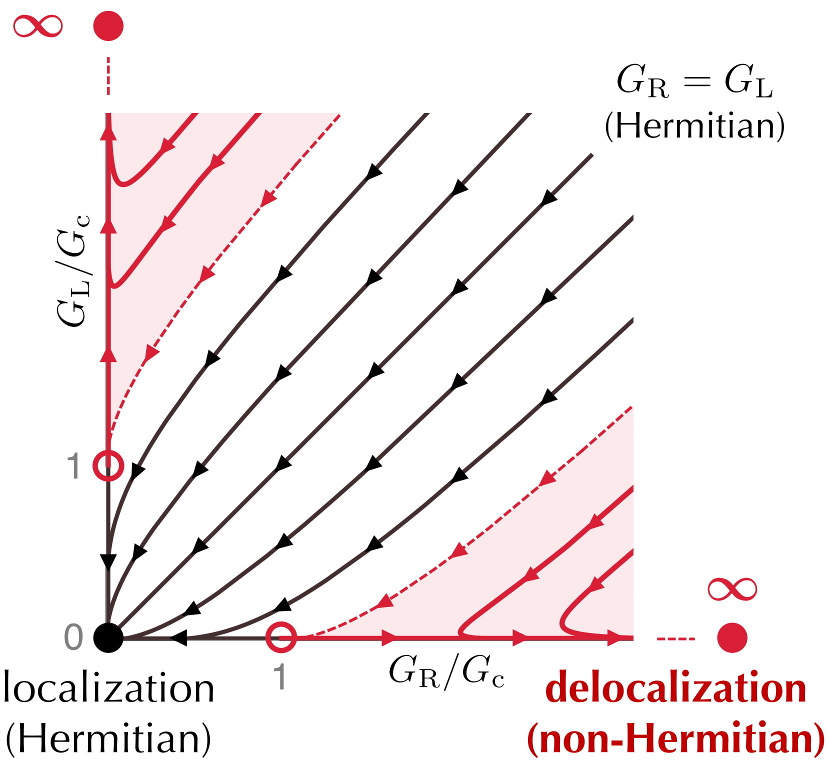
Around $\gamma = \gamma_{\mathrm{c}} := 1/\ell$, G_{R} exhibits criticality.

$$|G_{\mathrm{R}} - G_{\mathrm{c}}| \propto |\gamma - \gamma_{\mathrm{c}}|$$

Two-parameter scaling

Scaling equation $\frac{\langle dT_{\mathrm{R/L}} \rangle}{dL} = \pm\gamma T_{\mathrm{R/L}} - \frac{T_{\mathrm{R/L}}\left(1 - R_{\mathrm{L/R}}\right)}{\ell}$

- Hermitian $(\gamma = 0)$: the scaling equation is described solely by L/ℓ
  (one-parameter scaling)
- Non-Hermitian $(\gamma \neq 0)$: **two length scales** ℓ, γ^{-1}



☆ **Two-parameter scaling** $(G_{\mathrm{R}} \,\&\, G_{\mathrm{L}})$

The unusual delocalization originates from the breakdown of the one-parameter scaling.

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Time reversal and reciprocity

☆ **Symmetry can change universality of Anderson localization.**

Threefold way by time-reversal symmetry (reciprocity)

Class A: no symmetry

Class AI: $H^* = H^T = H$

Class AII: $\sigma_2 H^* \sigma_2 = \sigma_2 H^T \sigma_2 = H$

- **Localization is enhanced in class AI, while it is suppressed in class AII.**

e.g. Delocalization in 2D class AII   Hikami *et al.*, PTP **63**, 707 (1980)

⟷ NO delocalization in 2D class A

Does symmetry change universality of localization even in non-Hermitian systems?

Reciprocity in non-Hermitian systems

☆ **Symmetry ramifies in non-Hermitian systems.**

Kawabata *et al.*, PRX **9**, 041015 (2019)

(Hermitian) (non-Hermitian)

$$H = H^* = H^T$$

→ $H = H^*$ **time-reversal symmetry**

→ $H = H^T$ **reciprocity**

Two symmetry is different in non-Hermitian systems, while it is equivalent in Hermitian systems.

- Why is the latter symmetry reciprocity?

$$H = H^T \longleftrightarrow S = S^T \longleftrightarrow t_{\mathrm{L}} = t_{\mathrm{R}}^T$$

(reciprocal transport)

☆ **Reciprocity leads to threefold universality of localization.**

No delocalization (class AI†)

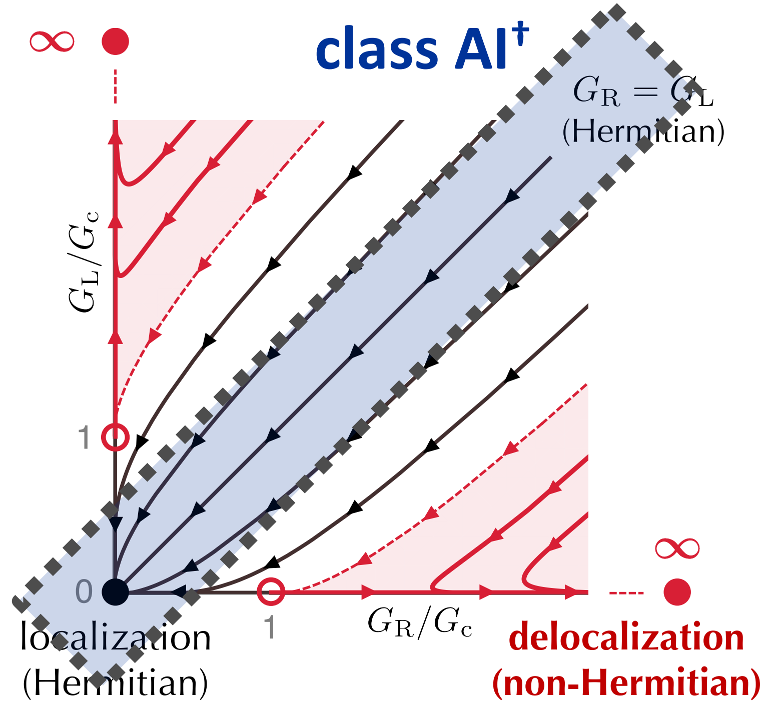
- Continuum model in class AI†

$$h_{\mathrm{AI}^\dagger} = -\mathrm{i}\tau_3\partial_x + m_0(x) + \left(m_1(x) + \mathrm{i}\gamma/2\right)\tau_1$$

reciprocity: $\tau_1 h_{\mathrm{AI}^\dagger}^T \tau_1 = h_{\mathrm{AI}^\dagger}$

⟶ $S = S^T,\ G_{\mathrm{R}} = G_{\mathrm{L}}$

⟶ **The universality is the same as the Hermitian counterpart.**



☆ The absence of delocalization in random lasers originates from reciprocity.

Hermitian Hamiltonian in class AI **+ complex onsite potential**

→ non-Hermitian Hamiltonian in class AI†

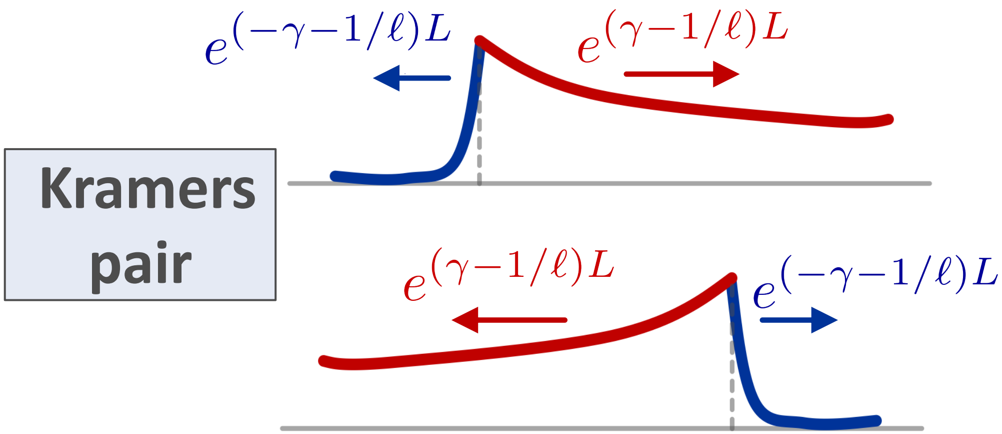
Bidirectional delocalization (class AII†)

- Continuum model in class AII†

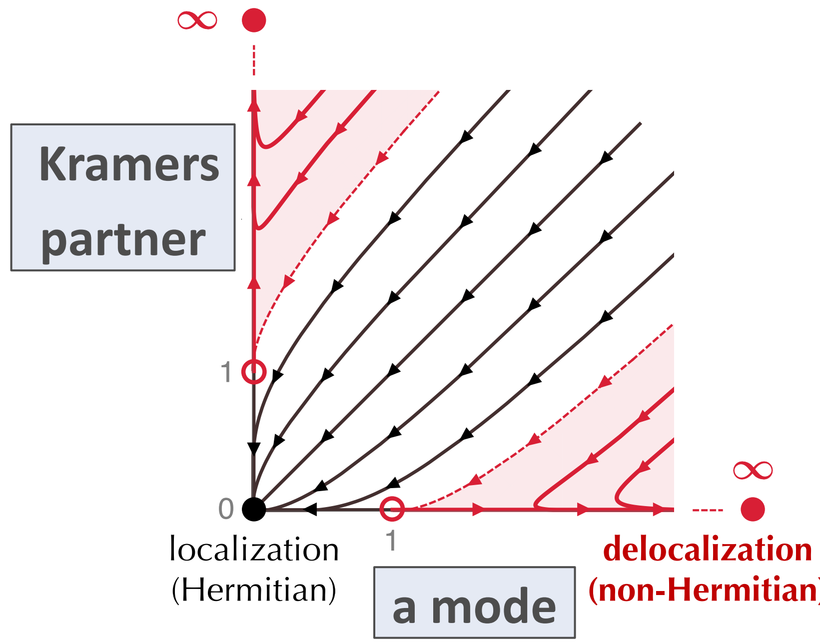
$$h_{\mathrm{AII}^\dagger} = \left(-\mathrm{i}\partial_x + \Delta\sigma_1 + \mathrm{i}\left(\gamma/2\right)\underbrace{\sigma_3}_{\text{spin}}\right)\tau_3 + m_0\left(x\right) + m_1\left(x\right)\underbrace{\tau_1}_{\text{orbit}}$$

reciprocity: $(\sigma_2\tau_1)\, h_{\mathrm{AII}^\dagger}^T\, (\sigma_2\tau_1) = h_{\mathrm{AII}^\dagger}$

→ **Kramers degeneracy**



$$G_{\mathrm{R}} = G_{\mathrm{L}} \sim e^{(\gamma-1/\ell)L} + e^{(-\gamma-1/\ell)L}$$



☆ Symplectic reciprocity gives rise to a new type of delocalization.

Threefold universality

☆ **Threefold universality of non-Hermitian localization by reciprocity**

Class	Symmetry	Delocalization	Conductances	
A	No	Unidirectional	$e^{(\pm\gamma-1/\ell)L}$	(Hatano-Nelson)
AI†	$H^T = H$	No	$e^{-L/\ell}$	(random laser)
AII†	$\sigma_2 H^T \sigma_2^{-1} = H$	Bidirectional	$e^{(\lvert\gamma\rvert-1/\ell)L}$	(symplectic Hatano-Nelson)

☆ **Localization is enhanced in class AI†, while it is suppressed in class AII†.**

Similar to the Hermitian counterparts,
but the influence pf reciprocity is more dramatic

☆ **Time-reversal symmetry does not change universality.**
$(H^* = H,\ \sigma_2 H^* \sigma_2 = H)$

Lattice models (class A)

Numerical calculations of localization lengths (**transfer-matrix method**).

e.g. Kramer *et al.*, Int. J. Mod. Phys. B **24**, 1841 (2010)

☆ **Two localization lengths** in non-Hermitian systems

(non-Hermitian) $\xi_{\mathrm{L}} \neq \xi_{\mathrm{R}},\ G_{\mathrm{L}} \neq G_{\mathrm{R}}$

(Hermitian) $\xi_{\mathrm{L}} = \xi_{\mathrm{R}},\ G_{\mathrm{L}} = G_{\mathrm{R}}$

- **Hatano-Nelson model (class A)**

$$\hat{H} = \sum_{n} \left\{ -\frac{1}{2} \left[\left(J + \frac{\gamma}{2} \right) \hat{c}_{n+1}^{\dagger} \hat{c}_{n} + \left(J - \frac{\gamma}{2} \right) \hat{c}_{n}^{\dagger} \hat{c}_{n+1} \right] + m_{n} \hat{c}_{n}^{\dagger} \hat{c}_{n} \right\}$$

Class	Symmetry	Delocalization	Conductances
A	No	Unidirectional	$e^{(\pm\gamma - 1/\ell)L}$
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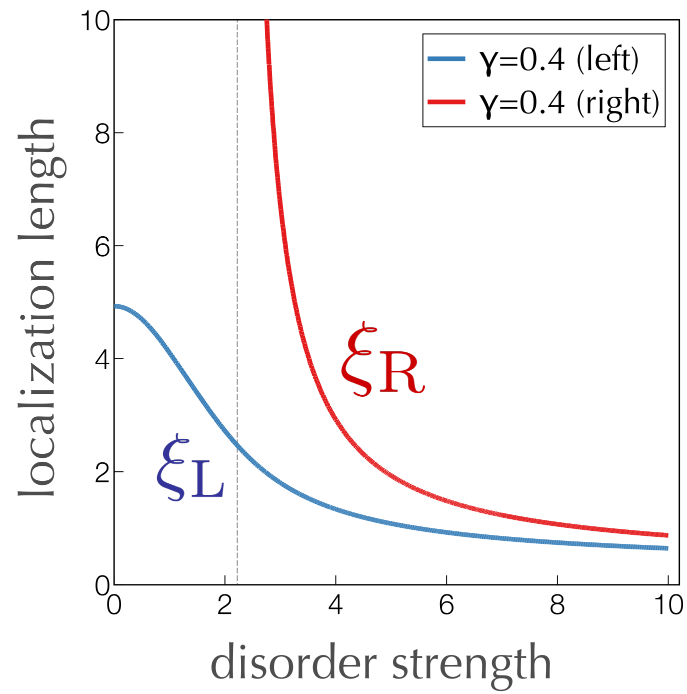

Lattice models (class AII†)

- **Symplectic Hatano-Nelson model (class AII†)** cf. Okuma *et al.*, PRL **124**, 086801 (2020)

$$\hat{H} = \sum_n \left\{ -\frac{1}{2} \left[\hat{c}_{n+1}^{\dagger} \left(J + \frac{\gamma\sigma_3}{2} - \mathrm{i}\Delta\sigma_1 \right) \hat{c}_n + \hat{c}_n^{\dagger} \left(J - \frac{\gamma\sigma_3}{2} + \mathrm{i}\Delta\sigma_1 \right) \hat{c}_{n+1} \right] + \hat{c}_n^{\dagger} \left(m_n + h\sigma_3 \right) \hat{c}_n \right\}$$

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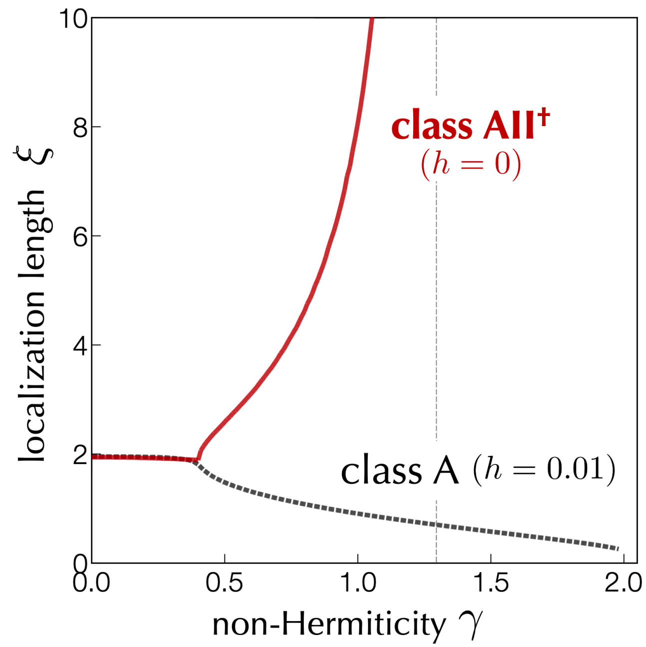


Reciprocity leads to $\xi_{\mathrm{L}} = \xi_{\mathrm{R}},\ G_{\mathrm{L}} = G_{\mathrm{R}}$

☆ **The delocalization is protected by reciprocity.**

A small symmetry-breaking perturbation destroys the delocalization.

$(h\sigma_z)$

cf. Okuma & Sato, PRL **123**, 097701 (2019)

Remarks

- **Other symmetry classes**

Dyson, PR **92**, 1331 (1953)

In the presence of chiral (sublattice) symmetry,
delocalization can appear even in Hermitian systems in 1D.

$$\tau_1 H \tau_1 = -H$$

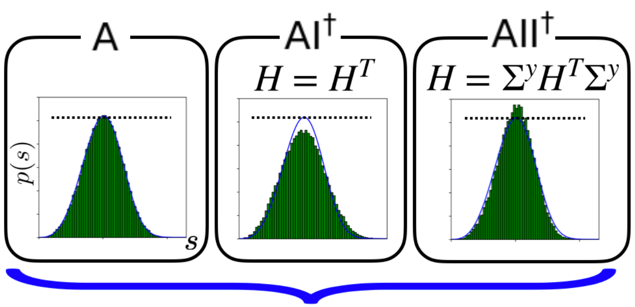
$$\tau_1 H \tau_1 = -H$$

nonunitary delocalization
(sublattice symmetry)

$$\tau_1 H^\dagger \tau_1 = -H$$

unitary delocalization
(chiral symmetry)

- **Random-matrix theory**



Three universal statistics

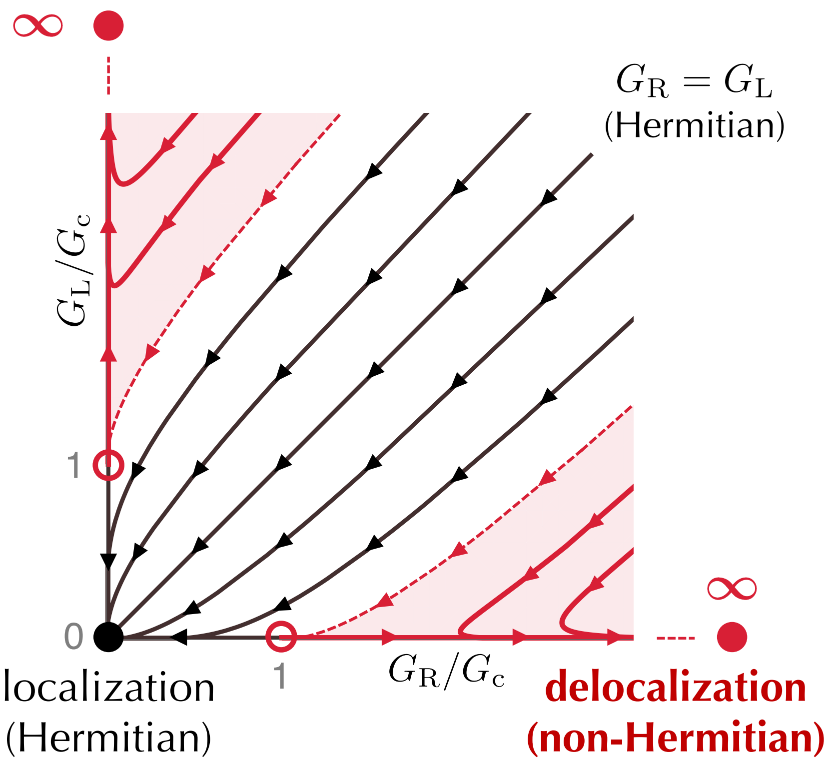
Reciprocity leads to **threefold universality of non-Hermitian random matrices**.

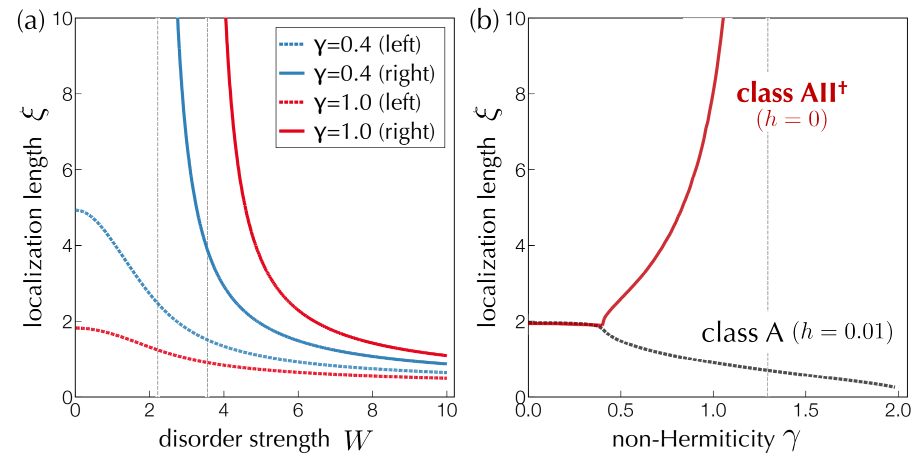
(Tomorrow's talk by Hamazaki)

Hamazaki, Kawabata, Kura & Ueda, PRR **2**, 023286 (2020)

Summary arXiv:2005.00604

- We develop a scaling theory of localization in non-Hermitian systems.
- Non-Hermiticity introduces a new scale and breaks down the one-parameter scaling: two-parameter scaling.
- Threefold universality based on reciprocity.





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