Asymptotic expansion

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26th April 2026

We consider the integral

$$Z = \sqrt{\frac{t}{\pi}} \int_{-\infty}^{\infty} d\phi \, e^{-H(\phi)}, \quad H(\phi) = t\phi^2 + u\phi^4 \quad (t > 0, u \geq 0), \tag{1}$$

which is normalized such that Z becomes unity for $u = 0$. This integral converges for $u \geq 0$. While it seems difficult to perform the integral analytically[*1], can we capture it perturbatively for sufficiently small u? Specifically, we may naively expand Z as

$$Z = \sum_{n=0}^{\infty} C_n u^n \tag{2}$$

with

$$C_n := \frac{(-1)^n}{n!} \sqrt{\frac{t}{\pi}} \int_{-\infty}^{\infty} d\phi \, \phi^{4n} e^{-t\phi^2} = \frac{(-1)^n}{\sqrt{\pi} t^{2n}} \frac{\Gamma(2n + 1/2)}{\Gamma(n+1)}. \tag{3}$$

For fixed n, the contribution $C_n u^n$ can be made small by taking u sufficiently small. However, for fixed u, $C_n u^n$ can grow rapidly for sufficiently large n, implying the breakdown of naive perturbation theory. Indeed, C_n asymptotically behaves as

$$C_n \sim \frac{1}{\sqrt{\pi n}} \left(-\frac{4n}{et^2} \right)^n \quad (n \to \infty). \tag{4}$$

Additionally, $|C_n u^n|$ decreases for $n \lesssim t^2/4u$, but increases for $n \gtrsim t^2/4u$.

In Fig. 1, we show the difference between the exact value of the integral and the perturbatively obtained value,

$$\Delta Z_n := \left| Z - \sum_{m=0}^{n} C_m u^m \right|, \tag{5}$$

as a function of the perturbative order n. While ΔZ_n decreases for $n < 25$, it instead increases for $n > 25$. This behavior is characteristic of an asymptotic expansion: the perturbative approximation improves up to an optimal order, beyond which higher-order terms worsen the approximation.

In passing, whereas the integral Z in Eq. (1) converges for $u \geq 0$, it diverges for $u < 0$. This implies that Z is not analytic around $u = 0$ [1].

[*1] Actually, Z is analytically evaluated with the modified Bessel function as

$$Z = \frac{t}{2\sqrt{\pi u}} e^{\frac{t^2}{8u}} K_{1/4}\left(\frac{t^2}{8u} \right).$$

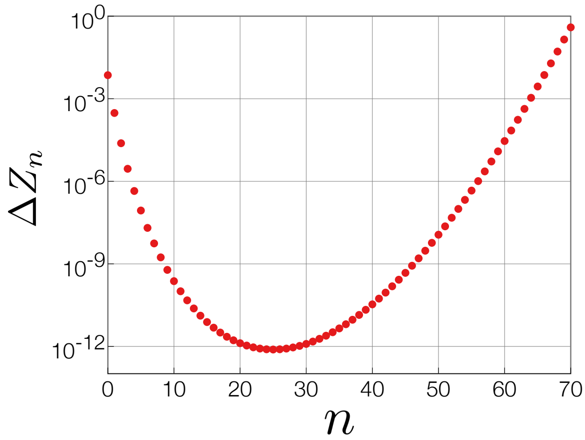

Fig. 1: ΔZ_n in Eq. (5) as a function of n for $t = 1.0$ and $u = 0.01$.

References

[1] F. J. Dyson, *Divergence of Perturbation Theory in Quantum Electrodynamics*, Phys. Rev. **85**, 631 (1952).