Locality in Landau-Ginzburg theory

Kohei Kawabata (Institute for Solid State Physics, University of Tokyo)

20th April 2026

Let us consider a generic two-body interaction,

$$U = \int d^d \boldsymbol{r} \int d^d \boldsymbol{r}' \, \phi(\boldsymbol{r}) \, K(\boldsymbol{r} - \boldsymbol{r}') \, \phi(\boldsymbol{r}'), \tag{1}$$

where $K = K(\boldsymbol{r})$ specifies details of the interaction (i.e., interaction kernel). Here, we assume that the interaction is invariant under rotation and that $K(\boldsymbol{r})$ depends only on $r := |\boldsymbol{r}|$. We expand the field as

$$\phi(\boldsymbol{r}') \simeq \phi(\boldsymbol{r}) + (\nabla \phi(\boldsymbol{r})) \cdot (\boldsymbol{r}' - \boldsymbol{r}) + \frac{1}{2} \sum_{i,j=1}^{d} (\partial_i \partial_j \phi(\boldsymbol{r})) (\boldsymbol{r}' - \boldsymbol{r})_i (\boldsymbol{r}' - \boldsymbol{r})_j , \tag{2}$$

and evaluate each contribution as follows:

- The first term gives

$$\begin{aligned} U_0 &= \int d^d \boldsymbol{r} \left[\left(\int d^d \boldsymbol{r}' \, K(\boldsymbol{r} - \boldsymbol{r}') \right) \phi(\boldsymbol{r})^2 \right] \\ &= \int d^d \boldsymbol{r} \left[\left(\int d^d \boldsymbol{r}' \, K(r') \right) \phi(\boldsymbol{r})^2 \right] \end{aligned} \tag{3}$$

- The second term gives

$$\begin{aligned} U_1 &= \int d^d \boldsymbol{r} \left[\left(\int d^d \boldsymbol{r}' \, (\boldsymbol{r}' - \boldsymbol{r}) \, K(\boldsymbol{r} - \boldsymbol{r}') \right) \cdot (\nabla \phi(\boldsymbol{r})) \, \phi(\boldsymbol{r}) \right] \\ &= \int d^d \boldsymbol{r} \left[\left(\int d^d \boldsymbol{r}' \, \boldsymbol{r}' K(r') \right) \cdot (\nabla \phi(\boldsymbol{r})) \, \phi(\boldsymbol{r}) \right] \\ &= 0, \end{aligned} \tag{4}$$

  where the last equality follows from rotation symmetry[*1]

- The third term gives

$$\begin{aligned} U_2 &= \frac{1}{2} \sum_{i,j=1}^{d} \int d^d \boldsymbol{r} \, \phi(\boldsymbol{r}) \, (\partial_i \partial_j \phi(\boldsymbol{r})) \int d^d \boldsymbol{r}' \, (\boldsymbol{r}' - \boldsymbol{r})_i (\boldsymbol{r}' - \boldsymbol{r})_j \, K(\boldsymbol{r} - \boldsymbol{r}') \\ &= \frac{1}{2} \sum_{i,j=1}^{d} \int d^d \boldsymbol{r} \, \phi(\boldsymbol{r}) \, (\partial_i \partial_j \phi(\boldsymbol{r})) \int d^d \boldsymbol{r}' \, (\boldsymbol{r}')_i (\boldsymbol{r}')_j \, K(r'). \end{aligned} \tag{5}$$

  Owing to rotation symmetry, we have

$$\int d^d \boldsymbol{r}' \, (\boldsymbol{r}')_i (\boldsymbol{r}')_j \, K(r') \propto \delta_{ij} \int d^d \boldsymbol{r}' \, (r')^2 \, K(r'). \tag{6}$$

[*1] This term also vanishes if we impose inversion symmetry [i.e., $K(\boldsymbol{r}) = K(-\boldsymbol{r})$].

Taking the trace of both sides, we obtain

$$\int d^d\boldsymbol{r}'\,(\boldsymbol{r}')_i\,(\boldsymbol{r}')_j\,K(r') = \frac{\delta_{ij}}{d}\int d^d\boldsymbol{r}'\,(r')^2\,K(r')\,. \tag{7}$$

Consequently, we have

$$\begin{aligned} U_2 &= \frac{1}{2d}\int d^d\boldsymbol{r}\left[\left(\int d^d\boldsymbol{r}'\,(r')^2\,K(r')\right)\phi(\boldsymbol{r})\left(\nabla^2\phi(\boldsymbol{r})\right)\right] \\ &= -\frac{1}{2d}\int d^d\boldsymbol{r}\left[\left(\int d^d\boldsymbol{r}'\,(r')^2\,K(r')\right)(\nabla\phi(\boldsymbol{r}))^2\right], \end{aligned} \tag{8}$$

where the boundary term is assumed to vanish.

Accordingly, U can be rewritten as

$$U \simeq \int d^d\boldsymbol{r}\left[t\phi^2 - \frac{\alpha}{2}(\nabla\phi)^2\right] \tag{9}$$

with

$$t := \int d^d\boldsymbol{r}\;K(r)\,, \quad \alpha := \frac{1}{d}\int d^d\boldsymbol{r}\;r^2 K(r)\,. \tag{10}$$

However, it is nontrivial whether the parameters obtained in Eq. (10) converge. The convergence of Eq. (10) depends on the long-distance behavior of the interaction, provided that it remains nonsingular at short distances. For short-range interactions, these parameters remain finite. For example, suppose that the interaction decays exponentially:

$$K(r) \propto e^{-r/\ell}, \tag{11}$$

where $\ell > 0$ denotes the characteristic interaction length scale. Then, the parameters in Eq. (10) are given by

$$t \propto \int_0^\infty dr\;r^{d-1}e^{-r/\ell} \propto \ell^d < \infty, \tag{12}$$

$$\alpha \propto \int_0^\infty dr\;r^{d+1}e^{-r/\ell} \propto \ell^{d+2} < \infty. \tag{13}$$

By contrast, for long-range interactions, the parameters in Eq. (10) can diverge. For example, suppose that the interaction only exhibits a power-law decay of the Coulomb type[*2]:

$$K(r) \propto \frac{1}{r^{d-2}}. \tag{14}$$

Then, the parameters in Eq. (10) behave as

$$t \propto \int_0^\infty dr\;r, \quad \alpha \propto \int_0^\infty dr\;r^3 \tag{15}$$

both of which diverge in the infinite-size limit. This implies that such long-range interactions may not, in general, be captured by a local derivative expansion.

[*2] For $d = 2$, we should instead consider $K(r) \propto \log r$.