

Homework II (Condensed Matter Physics II)

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We consider the following continuum Dirac model in two dimensions,

$$H(\mathbf{k}) = \begin{pmatrix} 0 & (k_x - ik_y)^2 \\ (k_x + ik_y)^2 & 0 \end{pmatrix} + m \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (\mathbf{k} \in \mathbb{R}^2, m \in \mathbb{R}). \quad (1)$$

This Hamiltonian effectively captures a valley degree of freedom in bilayer graphene [see, for example, E. McCann and V. I. Fal'ko, *Phys. Rev. Lett.* **96**, 086805 (2006); M. Koshino and T. Ando, *Phys. Rev. B* **73**, 245403 (2006)] with the mass term representing, for example, an externally applied electric field $m \in \mathbb{R}$. Our goal is to compute the Chern number of this model, which physically describes the valley Hall effect [see, for example, M. Sui *et al.*, *Nat. Phys.* **11**, 1027 (2015); Y. Shimazaki *et al.*, *Nat. Phys.* **11**, 1032 (2015)].

(1) Calculate the eigenenergies.

(2) We express the lower level $|\downarrow\rangle$ (i.e., eigenstate with the negative eigenenergy) by

$$|\downarrow\rangle = \begin{pmatrix} e^{-i\phi} \sin(\theta/2) \\ -\cos(\theta/2) \end{pmatrix} \quad (\phi \in [0, 2\pi), \theta \in [0, \pi]). \quad (2)$$

Determine ϕ and θ in terms of $\mathbf{k}$ and m .

(3) Calculate the Berry connection of the lower band $|\downarrow\rangle$ in a certain basis.

(4) Calculate the Berry curvature of the lower band $|\downarrow\rangle$.

(5) Calculate the Chern number of the lower band $|\downarrow\rangle$.