Homework I (Condensed Matter Physics II)

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The Hamiltonian of the Su-Schrieffer-Heeger model reads

$$\hat{H} = \sum_{n=1}^{L} \left[v \left(\hat{b}_n^\dagger \hat{a}_n + \hat{a}_n^\dagger \hat{b}_n \right) + t \left(\hat{a}_{n+1}^\dagger \hat{b}_n + \hat{b}_n^\dagger \hat{a}_{n+1} \right) \right], \tag{1}$$

where $\hat{a}_n$ and $\hat{b}_n$ ($\hat{a}_n^\dagger$ and $\hat{b}_n^\dagger$) annihilate (create) fermions on the two sublattices, and v (t) is the intracell (intercell) hopping amplitude. While we impose $\hat{a}_{L+1} = \hat{a}_1$ and $\hat{a}_{L+1}^\dagger = \hat{a}_1^\dagger$ under the periodic boundary conditions, we impose $\hat{a}_{L+1} = 0$ and $\hat{a}_{L+1}^\dagger = 0$ under the open boundary conditions. In the following, we assume $v, t > 0$ for simplicity.

(1) Perform the Fourier transform and write down the Bloch Hamiltonian $H(k)$. Confirm that the obtained Bloch Hamiltonian $H(k)$ respects chiral symmetry.

(2) Calculate the energy dispersion $E(k)$ and obtain the (single-particle) energy gap $\Delta \geq 0$.

(3) In each gapped phase, calculate the topological invariant (i.e., winding number) and make a phase diagram with respect to v/t.

(4) Determine a pair of zero modes in the topological phase under the semi-infinite boundary conditions. Specifically, suppose that the zero modes localized at the left and right edges are respectively described by ($\hat{H} \ket{0_{\mathrm{L}}} = \hat{H} \ket{0_{\mathrm{R}}} = 0$)

$$\ket{0_{\mathrm{L}}} = \sum_{n=1}^{\infty} \left(A_n^{\mathrm{L}} \hat{a}_n^\dagger + B_n^{\mathrm{L}} \hat{b}_n^\dagger \right) \ket{\mathrm{vac}}, \tag{2}$$

$$\ket{0_{\mathrm{R}}} = \sum_{n=1}^{\infty} \left(A_{L+1-n}^{\mathrm{R}} \hat{a}_{L+1-n}^\dagger + B_{L+1-n}^{\mathrm{R}} \hat{b}_{L+1-n}^\dagger \right) \ket{\mathrm{vac}} \tag{3}$$

with the fermionic vacuum state $\ket{\mathrm{vac}}$, and determine the coefficients $A_n^{\mathrm{L/R}}, B_n^{\mathrm{L/R}} \in \mathbb{C}$. You should normalize the zero modes by $\langle 0_{\mathrm{L}} | 0_{\mathrm{L}} \rangle = \langle 0_{\mathrm{R}} | 0_{\mathrm{R}} \rangle = 1$.

(5) Obtain the localization length ξ of the zero modes and express it by t and the energy gap Δ. Verify the divergence of ξ upon the gap closing.

The following assignment is optional. If you aim to achieve a higher grade, please complete it. However, if you are simply interested in getting a course credit, you may skip it.

(6) (optional) The wave functions in Eqs. (2) and (3) focus only on each edge and neglect the influence of the opposite edge (i.e., semi-infinite boundary conditions). In a finite-size system under the open boundary conditions, the two zero modes at the left and right edges are slightly coupled with each other through the bulk. To evaluate this finite-size effect, obtain the effective Hamiltonian within the zero-mode subspace

$$\hat{H}_{\mathrm{eff}} \coloneqq \begin{pmatrix} \langle 0_{\mathrm{L}} | \hat{H} | 0_{\mathrm{L}} \rangle & \langle 0_{\mathrm{L}} | \hat{H} | 0_{\mathrm{R}} \rangle \\ \langle 0_{\mathrm{R}} | \hat{H} | 0_{\mathrm{L}} \rangle & \langle 0_{\mathrm{R}} | \hat{H} | 0_{\mathrm{R}} \rangle \end{pmatrix} \tag{4}$$

and calculate its eigenenergies. If you are interested, you can compare your results with the numerical results.

(7) (optional) The zero modes are robust against disorder as long as the energy gap and chiral symmetry are preserved. To confirm this robustness, let us consider the disordered Su-Schrieffer-Heeger model,

$$\hat{H} = \sum_{n=1}^{L} \left[v_n \left(\hat{b}_n^{\dagger} \hat{a}_n + \hat{a}_n^{\dagger} \hat{b}_n \right) + t_n \left(\hat{a}_{n+1}^{\dagger} \hat{b}_n + \hat{b}_n^{\dagger} \hat{a}_{n+1} \right) \right], \tag{5}$$

with the site-dependent hopping amplitudes v_n and t_n ($n = 1, 2, \cdots, L$). Determine the zero mode localized at the left edge under the semi-infinite boundary conditions in a similar manner to Eq. (2). You do not have to consider the normalization conditions.