Classifying space and Clifford algebra[*1]

Kohei Kawabata (Institute for Solid State Physics, University of Tokyo)

18th January 2025

1 Classifying space

We summarize derivations of the classifying spaces for Hermitian (random) matrices in the tenfold Altland-Zirnbauer (AZ) symmetry classification (Table 1). The AZ symmetry consists of time-reversal symmetry, particle-hole symmetry, and chiral (sublattice) symmetry,

$$\mathcal{T} H \mathcal{T}^{-1} = H, \quad \mathcal{T}^2 = \pm 1, \tag{1}$$

$$\mathcal{C} H \mathcal{C}^{-1} = -H, \quad \mathcal{C}^2 = \pm 1, \tag{2}$$

$$\mathcal{S} H \mathcal{S}^{-1} = -H, \quad \mathcal{S}^2 = 1, \tag{3}$$

with antiunitary matrices $\mathcal{T}$ and $\mathcal{C}$, and a unitary matrix $\mathcal{S}$.

Below, we consider a flattened gapped Hermitian matrix H satisfying $H^2 = 1$. Specifically, for a given gapped Hermitian Hamiltonian,

$$H = \sum_{i \in \{\text{occupied}\}} E_i \left| \varphi_i \right\rangle \left\langle \varphi_i \right| + \sum_{i \in \{\text{empty}\}} E_i \left| \varphi_i \right\rangle \left\langle \varphi_i \right| \tag{4}$$

Table 1: Classifying spaces of Hermitian matrices in the tenfold Altland-Zirnbauer (AZ) symmetry classification. The AZ symmetry consists of time-reversal symmetry (TRS), particle-hole symmetry (PHS), and chiral symmetry (CS).

AZ class	TRS	PHS	CS		Classifying space	$\pi_0(\star)$
A	0	0	0	$\mathcal{C}_0$	$\mathrm{U}(m+n)/\mathrm{U}(m)\times\mathrm{U}(n)$	$\mathbb{Z}$
AIII	0	0	1	$\mathcal{C}_1$	$\mathrm{U}(n)$	0
AI	$+1$	0	0	$\mathcal{R}_0$	$\mathrm{O}(m+n)/\mathrm{O}(m)\times\mathrm{O}(n)$	$\mathbb{Z}$
BDI	$+1$	$+1$	1	$\mathcal{R}_1$	$\mathrm{O}(n)$	$\mathbb{Z}_2$
D	0	$+1$	0	$\mathcal{R}_2$	$\mathrm{O}(2n)/\mathrm{U}(n)$	$\mathbb{Z}_2$
DIII	-1	$+1$	1	$\mathcal{R}_3$	$\mathrm{U}(2n)/\mathrm{Sp}(n)$	0
AII	-1	0	0	$\mathcal{R}_4$	$\mathrm{Sp}(m+n)/\mathrm{Sp}(m)\times\mathrm{Sp}(n)$	$\mathbb{Z}$
CII	-1	-1	1	$\mathcal{R}_5$	$\mathrm{Sp}(n)$	0
C	0	-1	0	$\mathcal{R}_6$	$\mathrm{Sp}(n)/\mathrm{U}(n)$	0
CI	$+1$	-1	1	$\mathcal{R}_7$	$\mathrm{U}(n)/\mathrm{O}(n)$	0

*1 This note is based on Appendices A and B of my master's thesis.

with $E_i < 0$ for $i \in \{\text{occupied}\}$ and $E_i > 0$ and $i \in \{\text{empty}\}$, we focus on the flattened counterpart

$$H = \sum_{i\in\{\text{occupied}\}} (-1) \left|\varphi_i\right\rangle \left\langle\varphi_i\right| + \sum_{i\in\{\text{empty}\}} (+1) \left|\varphi_i\right\rangle \left\langle\varphi_i\right| . \tag{5}$$

In the absence of time-reversal symmetry with the sign -1 (i.e., classes A, AIII, AI, BDI, D, C, and CI), the numbers of occupied and empty bands are chosen to be n and m, respectively. Meanwhile, in classes DIII, AII, and CII, time-reversal symmetry with the sign -1 enforces Kramers degeneracy, and the numbers of occupied and empty bands are assumed to be $2n$ and $2m$, respectively. Furthermore, in the presence of particle-hole or chiral symmetry (i.e., classes AIII, BDI, D, DIII, CII, C, and CI), we should consider half filling $m = n$ so that the symmetry will be satisfied and the gap will be open.

1.1 Standard (Wigner-Dyson) class (classes A, AI, and AII)

We diagonalize the flattened Hermitian matrix H in Eq. (5) as

$$H = U \begin{pmatrix} I_m & 0 \\ 0 & -I_n \end{pmatrix} U^{-1}, \tag{6}$$

where I_n is the $n \times n$ identity matrix, and U diagonalizes H and belongs to

$$U \in \begin{cases} \mathrm{U}\left(m+n\right) & (\text{class A}) \,; \\ \mathrm{O}\left(m+n\right) & (\text{class AI}) \,; \\ \mathrm{Sp}\left(m+n\right) & (\text{class AII}) \,. \end{cases} \tag{7}$$

Here, $\mathrm{U}\left(m+n\right)$, $\mathrm{O}\left(m+n\right)$, and $\mathrm{Sp}\left(m+n\right)$ are unitary, orthogonal, and (compact) symplectic groups. Additionally, U follows the gauge transformation,

$$U \mapsto U \begin{pmatrix} \tilde{U}_m & 0 \\ 0 & \tilde{U}_n \end{pmatrix}, \quad \tilde{U}_i \in \begin{cases} \mathrm{U}\left(i\right) & (\text{class A}) \,; \\ \mathrm{O}\left(i\right) & (\text{class AI}) \,; \\ \mathrm{Sp}\left(i\right) & (\text{class AII}) \,. \end{cases} \tag{8}$$

Thus, the classifying spaces are given as the complex, real, and quaternionic Grassmannians:

$$\mathcal{C}_0 = \frac{\mathrm{U}\left(m+n\right)}{\mathrm{U}\left(m\right) \times \mathrm{U}\left(n\right)} \quad (\text{class A}) \,, \tag{9}$$

$$\mathcal{R}_0 = \frac{\mathrm{O}\left(m+n\right)}{\mathrm{O}\left(m\right) \times \mathrm{O}\left(n\right)} \quad (\text{class AI}) \,, \tag{10}$$

$$\mathcal{R}_4 = \frac{\mathrm{Sp}\left(m+n\right)}{\mathrm{Sp}\left(m\right) \times \mathrm{Sp}\left(n\right)} \quad (\text{class AII}) \,. \tag{11}$$

The $\mathbb{Z}$ topological invariant is given as the number n of the occupied bands (i.e., zeroth Chern number).

1.2 Chiral class (classes AIII, BDI, and CII)

Let us choose the unitary matrix $\mathcal{S}$ for chiral symmetry in Eq. (3) as $\mathcal{S} = \sigma_z \otimes I_n$ ($\mathcal{S} = \sigma_z \otimes I_{2n}$) in classes AIII and BDI (class CII). Then, the flattened Hermitian matrix H in Eq. (5) reads

$$H = \begin{pmatrix} 0 & h \\ h^{\dagger} & 0 \end{pmatrix}, \tag{12}$$

where h is an $n \times n$ ($2n \times 2n$) non-Hermitian matrix in classes AIII and BDI (class CII), and belongs to

$$h \in \begin{cases} \mathcal{C}_1 = \mathrm{U}\left(n\right) & \left(\text{class AIII}\right); \\ \mathcal{R}_1 = \mathrm{O}\left(n\right) & \left(\text{class BDI}\right); \\ \mathcal{R}_5 = \mathrm{Sp}\left(n\right) & \left(\text{class CII}\right). \end{cases} \tag{13}$$

In class BDI, the $\mathbb{Z}_2$ topological invariant $\nu \in \{0, 1\}$ is given by

$$\left(-1\right)^{\nu} \coloneqq \mathrm{sgn} \det h. \tag{14}$$

1.3 Bogoliubov-de Gennes class (classes D, DIII, C, and CI)

Classes D and C.—In class D, let us choose the antiunitary matrix $\mathcal{C}$ in Eq. (2) as $\mathcal{C} = I_{2n}\mathcal{K}$ with complex conjugation $\mathcal{K}$. Since $\mathrm{i}H$ is a real antisymmetric matrix, we diagonalize the flattened Hermitian matrix H in Eq. (5) with a proper basis as

$$H = \mathrm{i}O \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} O^{-1}, \tag{15}$$

where O is a $2n \times 2n$ orthogonal matrix:

$$O \in \mathrm{O}\left(2n\right). \tag{16}$$

This orthogonal matrix O obeys the gauge transformation $O \mapsto O\tilde{O}$ satisfying

$$\tilde{O} \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \tilde{O}^{-1} = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}, \quad \tilde{O} \in \mathrm{O}\left(2n\right). \tag{17}$$

When we introduce a matrix G that transforms $\sigma_z \otimes I_n$ to $\sigma_y \otimes I_n$, i.e.,

$$\sigma_y \otimes I_n = G\left(\sigma_z \otimes I_n\right) G^{-1}, \quad G \coloneqq \frac{1}{\sqrt{2}} \begin{pmatrix} I_n & -\mathrm{i}I_n \\ \mathrm{i}I_n & I_n \end{pmatrix}, \tag{18}$$

the above gauge transformation reduces to

$$(G^{-1}\tilde{O}G) \begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix} (G^{-1}\tilde{O}G)^{-1} = \begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix}. \tag{19}$$

Hence, the allowed gauge transformation is generally given by

$$\tilde{O} = G \begin{pmatrix} W & 0 \\ 0 & W^* \end{pmatrix} G^{-1}, \quad W \in \mathrm{U}(n). \tag{20}$$

Thus, the classifying space is given as

$$\mathcal{R}_2 = \frac{\mathrm{O}(2n)}{\mathrm{U}(n)} \quad (\text{class D}). \tag{21}$$

The $\mathbb{Z}_2$ topological invariant $\nu \in \{0, 1\}$ is given by

$$(-1)^{\nu} := \mathrm{sgn}\,\mathrm{Pf}\,(\mathrm{i}H). \tag{22}$$

In class C, let us choose the antiunitary matrix $\mathcal{C}$ in Eq. (2) as $\mathcal{C} = (\sigma_y \otimes I_n)\,\mathcal{K}$. Owing to particle-hole symmetry, we diagonalize the flattened Hermitian matrix H in Eq. (5) with a proper basis as

$$H = \mathrm{i}U \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} U^{-1}, \tag{23}$$

where U is a $2n \times 2n$ symplectic matrix:

$$U \in \mathrm{Sp}(n). \tag{24}$$

Since this symplectic matrix U has gauge ambiguity in a similar manner to class D, the classifying space is given as

$$\mathcal{R}_6 = \frac{\mathrm{Sp}(n)}{\mathrm{U}(n)} \quad (\text{class C}). \tag{25}$$

Classes CI and DIII.—In class CI, let us choose the unitary matrix $\mathcal{S}$ for chiral symmetry in Eq. (3) as $\mathcal{S} = \sigma_z \otimes I_n$ and the antiunitary matrix $\mathcal{T}$ for time-reversal symmetry in Eq. (1) as $\mathcal{T} = (\sigma_x \otimes I_n)\,\mathcal{K}$. Then, the flattened Hermitian matrix H in Eq. (5) reads Eq. (12), where the $n \times n$ non-Hermitian matrix h satisfies

$$h^T = h. \tag{26}$$

Here, h is generally expressed as

$$h = f^T f, \quad f \in \mathrm{U}(n), \tag{27}$$

and has the following gauge ambiguity:

$$f \mapsto gf, \quad g \in \mathrm{O}(n). \tag{28}$$

Thus, the classifying space is given as

$$\mathcal{R}_7 = \frac{\mathrm{U}(n)}{\mathrm{O}(n)} \quad (\text{class CI}). \tag{29}$$

In class DIII, let us choose the unitary matrix $\mathcal{S}$ for chiral symmetry in Eq. (3) as $\mathcal{S} = \sigma_z \otimes I_{2n}$ and the antiunitary matrix $\mathcal{T}$ for time-reversal symmetry in Eq. (1) as $\mathcal{T} = (\sigma_x \otimes \sigma_y \otimes I_n)\,\mathcal{K}$. Then, the flattened Hermitian matrix H in Eq. (5) reads Eq. (12), where the $2n \times 2n$ non-Hermitian matrix h satisfies

$$\left(\sigma_y \otimes I_n\right) h^T \left(\sigma_y \otimes I_n\right)^{-1} = h. \tag{30}$$

Here, h is generally expressed as

$$h = f^T \left(\sigma_y \otimes I_n\right) f \left(\sigma_y \otimes I_n\right), \quad f \in \mathrm{U}\left(2n\right), \tag{31}$$

and has the following gauge ambiguity:

$$f \mapsto gf, \quad g \in \mathrm{Sp}\left(n\right). \tag{32}$$

Thus, the classifying space is given as

$$\mathcal{R}_3 = \frac{\mathrm{U}\left(2n\right)}{\mathrm{Sp}\left(n\right)} \quad \left(\text{class DIII}\right). \tag{33}$$

2 Clifford algebra

We summarize the extension problem of Clifford algebra for all the AZ symmetry classes (Table 2). We consider a generic Hermitian Dirac Hamiltonian in d dimensions:

$$H\left(\boldsymbol{k}\right) = \sum_{i=1}^{d} k_i \Gamma_i + m\Gamma_0, \tag{34}$$

where $\boldsymbol{k} \coloneqq \left(k_1, \cdots, k_d\right)$ is the momentum deviation from a relevant point, and $\Gamma_0, \Gamma_1, \cdots, \Gamma_d$ form the Clifford relation:

$$\left\{\Gamma_i, \Gamma_j\right\} = 2\delta_{ij}. \tag{35}$$

In the presence of the AZ symmetry, the Dirac matrices respect

$$\mathcal{T}\,\Gamma_0\,\mathcal{T} = \Gamma_0, \quad \mathcal{T}\,\Gamma_i\,\mathcal{T} = -\Gamma_i \quad \left(i \neq 0\right); \tag{36}$$

$$\mathcal{C}\,\Gamma_0\,\mathcal{C} = -\Gamma_0, \quad \mathcal{C}\,\Gamma_i\,\mathcal{C} = \Gamma_i \quad \left(i \neq 0\right); \tag{37}$$

$$\mathcal{S}\,\Gamma_0\,\mathcal{S} = -\Gamma_0, \quad \mathcal{S}\,\Gamma_i\,\mathcal{S} = -\Gamma_i \quad \left(i \neq 0\right), \tag{38}$$

where $\mathcal{T}$ and $\mathcal{C}$ are antiunitary matrices for time-reversal and particle-hole symmetries, respectively, and $\mathcal{S}$ is a unitary matrix for chiral symmetry.

Complex Clifford algebra Cl_n is defined with a set of generators $\left\{e_i\right\}_{i=1,\cdots,n}$ that satisfies

$$\left\{e_i, e_j\right\} = 2\delta_{ij}. \tag{39}$$

This algebra is complex since these generators can be represented by complex matrices. Complex Clifford algebra satisfies the following formulas [1]:

$$Cl_1 \cong \mathbb{C} \oplus \mathbb{C}, \tag{40}$$

$$Cl_2 \cong \mathbb{C}\left(2\right), \tag{41}$$

$$Cl_{n+2} \cong Cl_n \otimes \mathbb{C}\left(2\right), \tag{42}$$

Table 2: Extension of Clifford algebra in the Altland-Zirnbauer (AZ) symmetry classification. The AZ symmetry consists of time-reversal symmetry (TRS), particle-hole symmetry (PHS), and chiral symmetry (CS). Spatial dimensions are denoted by d.

AZ class	TRS	PHS	CS		Extension
A	0	0	0	$\mathcal{C}_d$	$Cl_d \to Cl_{d+1}$
AIII	0	0	1	$\mathcal{C}_{d+1}$	$Cl_{d+1} \to Cl_{d+2}$
AI	$+1$	0	0	$\mathcal{R}_{-d}$	$Cl_{0,d+2} \to Cl_{1,d+2}$
BDI	$+1$	$+1$	1	$\mathcal{R}_{1-d}$	$Cl_{d+1,2} \to Cl_{d+1,3}$
D	0	$+1$	0	$\mathcal{R}_{2-d}$	$Cl_{d,2} \to Cl_{d,3}$
DIII	-1	$+1$	1	$\mathcal{R}_{3-d}$	$Cl_{d,3} \to Cl_{d,4}$
AII	-1	0	0	$\mathcal{R}_{4-d}$	$Cl_{2,d} \to Cl_{3,d}$
CII	-1	-1	1	$\mathcal{R}_{5-d}$	$Cl_{d+3,0} \to Cl_{d+3,1}$
C	0	-1	0	$\mathcal{R}_{6-d}$	$Cl_{d+2,0} \to Cl_{d+2,1}$
CI	$+1$	-1	1	$\mathcal{R}_{7-d}$	$Cl_{d+2,1} \to Cl_{d+2,2}$

where $\mathbb{C}(2)$ is a fixed representation for 2×2 complex matrices. As we show below, the classifying space $\mathcal{C}_n$ corresponds to the extension problem $Cl_n \to Cl_{n+1}$. Since $\mathbb{C}(2)$ does not affect the extension problem, the above formulas lead to a periodic structure of the classifying space (i.e., Bott periodicity for the complex AZ class):

$$\mathcal{C}_{n+2} \cong \mathcal{C}_n. \tag{43}$$

Real Clifford algebra $Cl_{p,q}$ is defined with a set of generators $\{e_i\}_{i=1,\cdots,n}$ that satisfies

$$\{e_i, e_j\} = 0 \ \ (i \neq j), \quad e_i^2 = \begin{cases} -1 & (i = 1, \cdots, p); \\ +1 & (i = p+1, \cdots, p+q). \end{cases} \tag{44}$$

This algebra is real since these generators can be represented by real matrices. Real Clifford algebra satisfies the following formulas [1]:

$$Cl_{0,1} \cong \mathbb{R} \oplus \mathbb{R}, \tag{45}$$
$$Cl_{0,2} \cong \mathbb{R}(2), \tag{46}$$
$$Cl_{1,0} \cong \mathbb{C}, \tag{47}$$
$$Cl_{2,0} \cong \mathbb{H}, \tag{48}$$
$$Cl_{p+1,q+1} \cong Cl_{p,q} \otimes \mathbb{R}(2), \tag{49}$$
$$Cl_{p,q} \otimes Cl_{0,2} \cong Cl_{q,p+2}, \tag{50}$$
$$Cl_{p,q} \otimes Cl_{2,0} \cong Cl_{q+2,p}, \tag{51}$$
$$Cl_{p,q} \otimes Cl_{0,4} \cong Cl_{p,q+4}, \tag{52}$$
$$Cl_{p+8,q} \cong Cl_{q,q+8} \cong Cl_{p,q} \otimes \mathbb{R}(16), \tag{53}$$

where $\mathbb{R}(n)$ is a fixed representation for $n \times n$ real matrices. As we show below, the classifying space $\mathcal{R}_{q-p}$ corresponds to the extension problem $Cl_{p,q} \to Cl_{p,q+1}$. Since $\mathbb{R}(16)$ does not affect the extension problem, the above formulas lead to a periodic structure of the classifying space (i.e., Bott periodicity for the real AZ class):

$$\mathcal{R}_{n+8} \cong \mathcal{R}_n. \tag{54}$$

1. *Class A.*—In the absence of symmetry, a set of operators

$$\{\Gamma_1, \cdots, \Gamma_d, \Gamma_0\} \tag{55}$$

forms complex Clifford algebra Cl_{d+1}. The extension problem reduces to $Cl_d \to Cl_{d+1}$.

2. *Class AIII.*—In the presence of chiral symmetry, a set of operators

$$\{\Gamma_1, \cdots, \Gamma_d, \mathcal{S}, \Gamma_0\} \tag{56}$$

forms complex Clifford algebra Cl_{d+2}. The extension problem reduces to $Cl_{d+1} \to Cl_{d+2}$.

3. *Class AI.*—A set of operators

$$\{J\Gamma_0; \mathcal{T}, J\mathcal{T}, \Gamma_1, \cdots, \Gamma_d\} \tag{57}$$

forms real Clifford algebra $Cl_{1,d+2}$, where J is a representation of the imaginary unit. The extension problem reduces to $Cl_{0,d+2} \to Cl_{1,d+2}$.

4. *Class BDI.*—A set of operators

$$\{J\mathcal{C}\mathcal{T}, J\Gamma_1, \cdots, J\Gamma_d; \mathcal{C}, J\mathcal{C}, \Gamma_0\} \tag{58}$$

forms real Clifford algebra $Cl_{d+1,3}$. The extension problem reduces to $Cl_{d+1,2} \to Cl_{d+1,3}$.

5. *Class D.*—A set of operators

$$\{J\Gamma_1, \cdots, J\Gamma_d; \mathcal{C}, J\mathcal{C}, \Gamma_0\} \tag{59}$$

forms real Clifford algebra $Cl_{d,3}$. The extension problem reduces to $Cl_{d,2} \to Cl_{d,3}$.

6. *Class DIII.*—A set of operators

$$\{J\Gamma_1, \cdots, J\Gamma_d; \mathcal{C}, J\mathcal{C}, J\mathcal{C}\mathcal{T}, \Gamma_0\} \tag{60}$$

forms real Clifford algebra $Cl_{d,4}$. The extension problem reduces to $Cl_{d,3} \to Cl_{d,4}$.

7. *Class AII.*—A set of operators

$$\{\mathcal{T}, J\mathcal{T}, J\Gamma_0; \Gamma_1, \cdots, \Gamma_d\} \tag{61}$$

forms real Clifford algebra $Cl_{3,d}$. The extension problem reduces to $Cl_{2,d} \to Cl_{3,d}$.

8. *Class CII.*—A set of operators

$$\{\mathcal{C}, J\mathcal{C}, J\mathcal{C}\mathcal{T}, J\Gamma_1, \cdots, J\Gamma_d; \Gamma_0\} \tag{62}$$

forms real Clifford algebra $Cl_{d+3,1}$. The extension problem reduces to $Cl_{d+3,0} \to Cl_{d+3,1}$.

9. *Class C*.—A set of operators

$$\{\mathcal{C},\ J\mathcal{C},\ J\Gamma_1,\ \cdots,J\Gamma_d;\ \Gamma_0\} \tag{63}$$

forms real Clifford algebra $Cl_{d+2,1}$. The extension problem reduces to $Cl_{d+2,0} \to Cl_{d+2,1}$.

10. *Class CI*.—A set of operators

$$\{\mathcal{C},\ J\mathcal{C},\ J\Gamma_1,\ \cdots,J\Gamma_d;\ J\mathcal{C}\mathcal{T},\ \Gamma_0\} \tag{64}$$

forms real Clifford algebra $Cl_{d+2,2}$. The extension problem reduces to $Cl_{d+2,1} \to Cl_{d+2,2}$.

References

[1] M. Karoubi, *K-Theory: An Introduction* (Springer, Berlin, Heidelberg, 1978).