東京大学 物性研究所
THE INSTITUTE FOR SOLID STATE PHYSICS
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Real class:
Z_2 topological invariants

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Classification of topological insulators

Periodic table of topological insulators and superconductors

AZ Symmetry				Dimension							
Class	TRS	PHS	CS	0	1	2	3	4	5	6	7
A	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AIII	0	0	1	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AI	+1	0	0	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
BDI	+1	+1	1	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$
D	0	+1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0
DIII	-1	+1	1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$
AII	-1	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
CII	-1	-1	1	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
C	0	-1	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
CI	+1	-1	1	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$

Schnyder, Ryu, Furusaki & Ludwig, PRB **78**, 195125 (2008)

Kitaev, AIP Conf. Proc. **1134**, 22 (2009)

Ryu, Schnyder, Furusaki & Ludwig, NJP **12**, 065010 (2010)

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(*d*/2)th Chern number

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AIII	0	0	1	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AI	+1	0	0	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
BDI	+1	+1	1	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$
D	0	+1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0
DIII	-1	+1	1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$
AII	-1	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
CII	-1	-1	1	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
C	0	-1	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
CI	+1	-1	1	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$

***d*D winding number**

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AIII	0	0	1	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AI	+1	0	0	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
BDI	+1	+1	1	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$
D	0	+1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$					
DIII	-1	+1	1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$
AII	-1	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
CII	-1	-1	1	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
C	0	-1	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
CI	+1	-1	1	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$

Fu-Kane invariant

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AZ Symmetry				Dimension							
Class	TRS	PHS	CS	0	1	2	3	4	5	6	7
A	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AIII	0	0	1								
AI	+1	0	0								
BDI	+1	+1	1								
D	0	+1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0
DIII	-1	+1	1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$
AII	-1	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
CII	-1	-1	1	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
C	0	-1	0	0							
CI	+1	-1	1	0							

Chern-Simons invariant (1-form)
[(time-reversal) polarization]

Chern-Simons invariant (3-form)

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Periodic table of topological insulators and superconductors

AZ Symmetry				Dimension							
Class	TRS	PHS	CS	0	1	2	3	4	5	6	7
A	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AIII	0	0	1	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AI	+1	0	0	$\mathbb{Z}$							$\mathbb{Z}_2$
BDI	+1	+1	1	$\mathbb{Z}_2$							$\mathbb{Z}_2$
D	0	+1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$					
DIII	-1	+1	1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$
AII	-1	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
CII	-1	-1	1	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
C	0	-1	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
CI	+1	-1	1	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$

Chern-Simons invariant

Fu-Kane invariant

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Topological invariants

TABLE III. Topological invariants for the tenfold Altland-Zirnbauer classes. There exist four types of invariants: For the $\mathbb{Z}$ indices, "W_d" is the d-dimensional winding number and "$\mathrm{Ch}_{d/2}$" is the $d/2$-th Chern number, which are defined in odd and even d spatial dimensions, respectively. For the $\mathbb{Z}_2$ indices, "$\mathrm{FK}_{d/2}$" denotes the Fu-Kane invariant in even d dimensions, and "CS_d ($\widetilde{\mathrm{CS}}_d$)" represents the Chern-Simons invariant in symmetry classes without (with) chiral symmetry in odd d dimensions. The entries "0" indicate the absence of nontrivial topological phases.

Class	TRS	PHS	CS	$d=1$	$d=2$	$d=3$	$d=4$	$d=5$	$d=6$	$d=7$	$d=8$
A	0	0	0	0	$\mathrm{Ch}_1 \in \mathbb{Z}^{\checkmark}$	0	$\mathrm{Ch}_2 \in \mathbb{Z}^{\checkmark}$	0	$\mathrm{Ch}_3 \in \mathbb{Z}^{\checkmark}$	0	$\mathrm{Ch}_4 \in \mathbb{Z}^{\checkmark}$
AIII	0	0	1	$W_1 \in \mathbb{Z}^{\times}$	0	$W_3 \in \mathbb{Z}^{\times}$	0	$W_5 \in \mathbb{Z}^{\times}$	0	$W_7 \in \mathbb{Z}^{\times}$	0
AI	+1	0	0	0	0	0	$\mathrm{Ch}_2 \in 2\mathbb{Z}^{\checkmark}$	0	$\mathrm{FK}_3 \in \mathbb{Z}_2^{\checkmark}$	$\mathrm{CS}_7 \in \mathbb{Z}_2^{\checkmark}$	$\mathrm{Ch}_4 \in \mathbb{Z}^{\checkmark}$
BDI	+1	+1	1	$W_1 \in \mathbb{Z}^{\times}$	0	0	0	$W_5 \in 2\mathbb{Z}^{\times}$	0	$\widetilde{\mathrm{CS}}_7 \in \mathbb{Z}_2^{\times}$	$\mathrm{FK}_4 \in \mathbb{Z}_2^{\times}$
D	0	+1	0	$\mathrm{CS}_1 \in \mathbb{Z}_2^{\times}$	$\mathrm{Ch}_1 \in \mathbb{Z}^{\checkmark}$	0	0	0	$\mathrm{Ch}_3 \in 2\mathbb{Z}^{\checkmark}$	0	$\mathrm{FK}_4 \in \mathbb{Z}_2^{\times}$
DIII	−1	+1	1	$\widetilde{\mathrm{CS}}_1 \in \mathbb{Z}_2^{\times}$	$\mathrm{FK}_1 \in \mathbb{Z}_2^{\checkmark}$	$W_3 \in \mathbb{Z}^{\checkmark/\times}$	0	0	0	$W_7 \in 2\mathbb{Z}^{\times}$	0
AII	−1	0	0	0	$\mathrm{FK}_1 \in \mathbb{Z}_2^{\checkmark}$	$\mathrm{CS}_3 \in \mathbb{Z}_2^{\checkmark}$	$\mathrm{Ch}_2 \in \mathbb{Z}^{\checkmark}$	0	0	0	$\mathrm{Ch}_4 \in 2\mathbb{Z}^{\checkmark}$
CII	−1	−1	1	$W_1 \in 2\mathbb{Z}^{\times}$	0	$\widetilde{\mathrm{CS}}_3 \in \mathbb{Z}_2^{\times}$	$\mathrm{FK}_2 \in \mathbb{Z}_2^{\times}$	$W_5 \in \mathbb{Z}^{\times}$	0	0	0
C	0	−1	0	0	$\mathrm{Ch}_1 \in 2\mathbb{Z}^{\checkmark}$	0	$\mathrm{FK}_2 \in \mathbb{Z}_2^{\times}$	$\mathrm{CS}_5 \in \mathbb{Z}_2^{\times}$	$\mathrm{Ch}_3 \in \mathbb{Z}^{\checkmark}$	0	0
CI	+1	−1	1	0	0	$W_3 \in 2\mathbb{Z}^{\times}$	0	$\widetilde{\mathrm{CS}}_5 \in \mathbb{Z}_2^{\times}$	$\mathrm{FK}_3 \in \mathbb{Z}_2^{\checkmark}$	$W_7 \in \mathbb{Z}^{\checkmark/\times}$	0

Shiozaki, Nakamura, Shimomura, Sato & Kawabata, arXiv:2407.18273