Homework III (Condensed Matter Physics II)

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Deadline: 20th January 2025

This is an optional assignment. If you aim to achieve a high grade, such as "A（優）", please solve it. However, if you are simply interested in getting a credit, you do not have to solve it.

In contrast to time-reversal-invariant topological insulators (i.e., class AII) in three dimensions, the spectrum of surface states in chiral-symmetric topological insulators (i.e., class AIII) in three dimensions can be detached from the bulk bands [A. Altland *et al.*, Phys. Rev. X **14**, 011057 (2024); D. Nakamura *et al.*, arXiv:2407.09458]. This phenomenon is a direct consequence of the absence of Wannier obstructions in the bulk bands. Here, let us explicitly see the detachment of Dirac surface states in a chiral-symmetric topological insulator in three dimensions, described by the following continuum model:

$$H(\boldsymbol{k}) = k_x \sigma_x + k_y \sigma_y. \tag{1}$$

(1) Confirm that the Dirac surface model in Eq. (1) respects chiral symmetry.

(2) Calculate the energy spectrum of the Dirac surface model in Eq. (1).

(3) The stability of the Dirac point $\boldsymbol{k} = 0$ is ensured by the topological invariant away from it. Specifically, this topological invariant is given as the one-dimensional winding number along a closed loop in momentum space encircling the Dirac point $\boldsymbol{k} = 0$. Calculate this topological invariant explicitly.

(4) Let us couple the Dirac surface model in Eq. (1) to a trivial band $H = \varepsilon \sigma_x$ ($\varepsilon \geq 0$) and study the four-band model,

$$\begin{aligned} H(k_x, k_y) &= \begin{pmatrix} k_x \sigma_x + k_y \sigma_y & v\sigma_- \\ v\sigma_+ & \varepsilon \sigma_x \end{pmatrix} \\ &= \frac{1}{2}\left((k_x + \varepsilon)\sigma_x + k_y \sigma_y\right) + \frac{1}{2}\left((k_x - \varepsilon)\sigma_x + k_y \sigma_y\right)\tau_z + \frac{v}{2}\left(\sigma_x \tau_x + \sigma_y \tau_y\right), \end{aligned} \tag{2}$$

where $v \geq 0$ denotes the coupling strength between the Dirac surface model and the trivial band. Calculate the energy spectrum of this four-band model. In the obtained energy spectrum, the surface band near zero energy should be detached from the bulk bands at infinity.

Hint: calculate $\left(H(k_x, k_y)\right)^2$.