Homework II (Condensed Matter Physics II)

Lecturer: Kohei Kawabata (Institute for Solid State Physics, University of Tokyo)

Deadline: 16th December 2024

Let us derive a chiral edge state in a Chern insulator described by the Bloch Hamiltonian

$$H\left(k_x, k_y\right) = \left(t \sin k_x\right) \sigma_x + \left(t \sin k_y\right) \sigma_y + \left(m + t \cos k_x + t \cos k_y\right) \sigma_z \tag{1}$$

with real parameters $t, m \in \mathbb{R}$ ($t \neq 0$) and Pauli matrices σ_i's ($i = x, y, z$).

(1) Calculate the energy dispersion $E\left(k_x, k_y\right)$ (under the periodic boundary conditions in both directions) and identify the conditions under which the band gap closes.

In the following, we place the system on a square lattice of length L, and impose the open boundary conditions along the x direction and the periodic boundary conditions along the y direction. Let $\vec{\psi}\left(x, k_y\right)$ be a two-component single-particle eigenstate at site x and momentum k_y. Then, the single-particle Schrödinger equation reads

$$T\left(k_y\right) \vec{\psi}\left(x-1, k_y\right) + M\left(k_y\right) \vec{\psi}\left(x, k_y\right) + T^{\dagger}\left(k_y\right) \vec{\psi}\left(x+1, k_y\right) = E\left(k_y\right) \vec{\psi}\left(x, k_y\right) \tag{2}$$

in the bulk ($x = 2, 3, \cdots, L-1$) and

$$M\left(k_y\right) \vec{\psi}\left(1, k_y\right) + T^{\dagger}\left(k_y\right) \vec{\psi}\left(2, k_y\right) = E\left(k_y\right) \vec{\psi}\left(1, k_y\right), \tag{3}$$

$$T\left(k_y\right) \vec{\psi}\left(L-1, k_y\right) + M\left(k_y\right) \vec{\psi}\left(L, k_y\right) = E\left(k_y\right) \vec{\psi}\left(L, k_y\right), \tag{4}$$

at the left ($x = 1$) and right ($x = L$) edges, respectively. Here, $E\left(k_y\right)$ is an eigenenergy for given k_y, and the two-by-two matrices $M\left(k_y\right)$ and $T\left(k_y\right)$ represent the mass and hopping terms, respectively.

(2) Determine explicit expressions for the two-by-two matrices $M\left(k_y\right)$ and $T\left(k_y\right)$ above.

To obtain a chiral edge state, we focus on an eigenstate localized at the left edge ($x = 1$) and assume an ansatz for the eigenstate of the form:

$$\vec{\psi}\left(x, k_y\right) \propto \left(\lambda\left(k_y\right)\right)^{x-1} \vec{v}\left(k_y\right), \tag{5}$$

where $\lambda\left(k_y\right)$ is a parameter that determines the localization length, and $\vec{v}\left(k_y\right)$ is a two-component vector that describes the internal degree of freedom.

(3) Substitute Eq. (5) into Eqs. (2) and (3), and then determine $\lambda\left(k_y\right)$ and $E\left(k_y\right)$.

(4) To have a normalizable eigenstate, $\lambda\left(k_y\right)$ should satisfy a certain condition. From this condition, derive a condition for the presence of the chiral edge state. Notably, this should coincide with the condition for the nontrivial Chern number in the bulk wave function.