Symmetry of the (generalized) Bernevig-Hughes-Zhang model

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Let us begin with a Chern insulator (Qi-Wu-Zhang model [1]),

$$H_{\mathrm{QWZ}}\left(\boldsymbol{k}\right) = \left(t \sin k_x\right) \tau_x + \left(t \sin k_y\right) \tau_y + \left(m + t \cos k_x + t \cos k_y\right) \tau_z \tag{1}$$

with $t, m \in \mathbb{R}$. This model respects particle-hole symmetry

$$\tau_x H_{\mathrm{QWZ}}^{*}\left(\boldsymbol{k}\right) \tau_x^{-1} = -H_{\mathrm{QWZ}}\left(-\boldsymbol{k}\right), \quad \left(\tau_x \mathcal{K}\right)^2 = +1, \tag{2}$$

with a Pauli matrix τ_x and complex conjugation $\mathcal{K}$, and thus belongs to class D. While the nonzero Chern number does not necessitate particle-hole symmetry, it is compatible with particle-hole symmetry.

Building on this Chern insulator, we construct a $\mathbb{Z}_2$ topological insulator (Bernevig-Hughes-Zhang model [2])

$$\begin{aligned} H_{\mathrm{BHZ}}\left(\boldsymbol{k}\right) &= \begin{pmatrix} H_{\mathrm{QWZ}}\left(\boldsymbol{k}\right) & \delta_y \tau_y \\ \delta_y \tau_y & H_{\mathrm{QWZ}}^{*}\left(-\boldsymbol{k}\right) \end{pmatrix}_{\sigma} \\ &= \left(t \sin k_x\right) \sigma_z \tau_x + \left(t \sin k_y\right) \tau_y + \left(m + t \cos k_x + t \cos k_y\right) \tau_z + \delta_y \sigma_x \tau_y \end{aligned} \tag{3}$$

with $\delta_y \in \mathbb{R}$. While Pauli matrices σ_i's ($i = x, y, z$) describe the spin degree of freedom, τ_i's describe the orbital degree of freedom. This model respects time-reversal symmetry

$$\sigma_y H_{\mathrm{BHZ}}^{*}\left(\boldsymbol{k}\right) \sigma_y^{-1} = H_{\mathrm{BHZ}}\left(-\boldsymbol{k}\right), \quad \left(\sigma_y \mathcal{K}\right)^2 = -1. \tag{4}$$

As a result of particle-hole symmetry in Eq. (2), $H_{\mathrm{BHZ}}\left(\boldsymbol{k}\right)$ also respects particle-hole symmetry:

$$\tau_x H_{\mathrm{BHZ}}^{*}\left(\boldsymbol{k}\right) \tau_x^{-1} = -H_{\mathrm{BHZ}}\left(-\boldsymbol{k}\right), \quad \left(\tau_x \mathcal{K}\right)^2 = +1. \tag{5}$$

The combination of time-reversal symmetry in Eq. (4) and particle-hole symmetry in Eq. (5) leads to chiral symmetry:

$$\left(\sigma_y \tau_x\right) H_{\mathrm{BHZ}}\left(\boldsymbol{k}\right) \left(\sigma_y \tau_x\right)^{-1} = -H_{\mathrm{BHZ}}\left(\boldsymbol{k}\right), \quad \left(\sigma_y \tau_x\right)^2 = 1. \tag{6}$$

Consequently, $H_{\mathrm{BHZ}}\left(\boldsymbol{k}\right)$ belongs to class DIII. The $\mathbb{Z}_2$ topological invariant for class DIII is isomorphic to that for class AII. Each symmetry imposes the following constraint on the energy spectrum:

- Time-reversal symmetry in Eq. (4) gives rise to $\left(E\left(\boldsymbol{k}\right), E\left(-\boldsymbol{k}\right)\right)$ pairs (i.e., Kramers pairs).
- Particle-hole symmetry in Eq. (5) gives rise to $\left(E\left(\boldsymbol{k}\right), -E\left(-\boldsymbol{k}\right)\right)$ pairs.
- Chiral symmetry in Eq. (6) gives rise to $\left(E\left(\boldsymbol{k}\right), -E\left(\boldsymbol{k}\right)\right)$ pairs.

Let us also add a perturbation:

$$V = \delta_z \sigma_z \tau_x \tag{7}$$

with $\delta_z \in \mathbb{R}$. While this perturbation breaks time-reversal symmetry in Eq. (4) and particle-hole symmetry in Eq. (5), it still respects chiral symmetry in Eq. (6) (i.e., class AIII). As a result, the spectrum remains symmetric with respect to zero energy, consistent with the numerical calculations.

References

[1] X.-L. Qi, Y.-S. Wu, and S.-C. Zhang, *Topological quantization of the spin Hall effect in two-dimensional paramagnetic semiconductors*, Phys. Rev. B **74**, 085308 (2006) [arXiv:cond-mat/0505308].

[2] B. A. Bernevig, T. L. Hughes, and S.-C. Zhang, *Quantum Spin Hall Effect and Topological Phase Transition in HgTe Quantum Wells*, Science **314**, 1757 (2006) [arXiv:cond-mat/0611399].